Online Appendices

Financial Innovation and the Transactions Demand for Cash

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Contents

A Statistics on the probability of cash thefts for Italy and the US

The statistics on the probability of cash thefts (κ) for Italy are computed in 3 steps.

(1) We consider 4 crimes where cash is lost: bag-snatching (Scippi), pickpocketing (borseggi), theft (furti), robbery (rapine). Using survey data on victimization per person (aged 14 or older) for the whole Italy (i.e. average of 103 provinces) in the year 2002 , gives the following percentages for each of the crimes, respectively:[1] 0.4 , 1.4 , 2.2, 0.3.

(2) Next, we adjust the statistic for each crime to take into account information on the percentage of crimes where cash is taken (source: Istat victimization survey). For instance for bag-snatching cash is taken 49% of the times. The statistics that we are interested in for 2002 is:

$$\kappa = (0.4 \cdot 0.49 + 1.4 \cdot 0.61 + 2.2 \cdot 0.37 + 0.3 \cdot 0.59) = 2.041$$

.

(3) Finally, using data on bag-snatching (scippo) and pickpocketing (borseggi) across 103 Italian provinces, and using a time series for these two crimes at the country level across years (source Istat), we construct values of κ that vary across provinces and years.

The statistics on victimization rates for the US are taken from Table 3 in "Rates of criminal victimization and percent change", relative to the years 1993 and 2005. We have the following crime rates:[2]

a) Personal Theft (see footnote *e*, includes pocket picking, completed bag snatching and attempted bag snatching), 0.23 in 1993 and 0.09 in 2005.

b) Robbery, completed property taken: 0.38 in 1993 and 0.17 in 2005.

c) Theft (completed) 23.01 in 1993 and 11.20 in 2005. Theft completed with known losses: 21.64 in 1993 and 10.20 in 2005

A.1 Three comparisons of the US vs. Italy

We first compare victimization rates in Italy and the US.

Table I: Victimization rates in Italy and the US

	Italy 2002	US 2005	US 1993
I) Sum of bag-snatching and pick-pocketing	1.8 %	0.09%	0.23%
II) Robbery	0.3%	0.17%	0.38%
III) Theft	2.2%	10.2%	21.64%

I and II are higher in Italy, but III is much higher in the US. If we apply the factor from Italy of the % of times that cash is taken to II and III, we obtain:

$$\begin{aligned}
\kappa_{IT}^{2002} &= (0.4 \cdot 0.49 + 1.4 \cdot 0.61 + 0.3 \cdot 0.59 + 2.2 \cdot 0.37) = 2.041 \\
\kappa_{US}^{2005} &= (0.09 + 0.17 \cdot 0.59 + 10.2 \cdot 0.37) = 3.96 \\
\kappa_{US}^{1993} &= (0.23 + 0.38 \cdot 0.59 + 21.64 \cdot 0.37) = 8.461
\end{aligned}$$

[1]Victimization rates: Istat, Figure 1.1 on page 13 of "La sicurezza dei cittadini, year 2002" (N. 18, published in 2004); Fraction of crimes where cash is taken: Istat Table 7.1 on page 67 of report "La sicurezza dei cittadini".

[2]per 100 persons (i.e. Percentage values), 12 years or older. Source: "Criminal Victimization Rates, 2002", http://www.ojp.usdoj.gov/bjs/pub/pdf/cv05.pdf

For bag snatching and pick-pocketing in the US we do not use a correction for the cash taken, since they are the rates of completed crime. This has a small influence on the figures since the victimization rates are small. The first line of the next Table summarizes these statistics

Table II: The probability of losing cash: Italy vs. US

	Italy	US
Our stats (κ) for 2002 - 2005	2.0	3.9
Stats from victimization survey		
Robbery (in 1991)	1.3	1.5
Theft of personal property (in 1991)	3.6	5.3
Stats from reports to the Police (in 2001)	1.3	1.9

In the second comparison we use data on victimization by the International Crime Victim Survey (ICVS), the most far reaching programme of standardized sample surveys to look a householders experience with crime.[3] The data are reported in the second and third line of table II. As before, the data (for 1991, a year when stats are available for both countries) have similar order of magnitude.

In the third comparison between the US and Italy we use data from reports to the police, as opposed to victimization rates.[4] The advantage is that perhaps these are more comparable. We note however that the definition of crimes is different from the one used in the victimization survey. In particular: the definition of theft for the police statistics is "stealing from a person with force or threat of force); includes muggings (bag-snatching) and theft with violence; excludes pickpocketing extortion and blackmailing. Note moreover that the police statistics may also include crimes on firms properties (not just household). The data for 2001 are reported in the fourth line of table II.

B Cash management and the household size

To study the effect of household size let's consider first two extreme cases. First suppose that a household is a collection of people, and that the data just collect their aggregates. Let s be the size of the household, i.e. the number of its members. Assume that b/c and p is common for all members. Under these assumptions:

$$\begin{aligned}(M/c)(s) &= (M/c)(1),\\ c(s) &= s\ c(1),\\ n(s) &= s\ n(1),\\ (W/M)(s) &= (W/M)(1)/s,\\ (\underline{M}/M)\ (s) &= (\underline{M}/M)\,(1)\,/s\ .\end{aligned}$$

At the other extreme suppose that a household behavior is completely centralized, independently of its

[3] Source: http://ruljis.leidenuniv.nl/group/jfcr/www/icvs/data/i_VIC.HTM

[4] The source is INTERNATIONAL COMPARISON OF CRIMINAL STATS, Table 1.4, year 2001 (http://www.homeoffice.gov.uk/rds/international1.html).

size. Then the size of of the household does not matter at all, so that:

$$\begin{aligned}(M/c)(s) &= (M/c)(1),\\ n(s) &= n(1),\\ (W/M)(s) &= (W/M)(1),\\ \left(\underline{M}/M\right)(s) &= \left(\underline{M}/M\right)(1).\end{aligned}$$

Table III: Cash management and Household size

Variable	ln c	ln n	ln M/c	ln W/M	ln $\underline{M}/M$	ln $\frac{n}{(c/2M)}$
Bivariate Regression						
ln (# adults in household)	0.50	0.20	-0.16	-0.13	-0.05	0.08
Multivariate Regression						
ln (# adults in household)	-	0.13	0.14			
ln c	-	0.18	-0.58			

We regress the (log of) each variable mentioned above on the (log of) the number of adults in the household. Each regression includes year, province, and ATM dummies, as well as the level of interest rates. The regressions are run at the household level, so that, depending on the variable there are approximately between 60,000 and 40,000 observations. The coefficient on household size is reported in table III:

Table IV: Cash management and Household size (alternative measures)

Variable	ln c	ln n	ln M/c
Bivariate Regressions			
ln (# family members)	0.43	0.20	-0.17
ln (# income receivers)	0.37	0.16	-0.09
Multivariate Regressions			
ln (# family members)	-	0.14	0.08
ln c	-	0.18	-0.57
ln (# income receivers)	-	0.10	0.12
ln c	-	0.20	-0.57

The estimated coefficients suggest that none of the extreme cases is a good approximations of the data. The coefficient of c suggests that there are economies of scale in the cash expenditure, since it is below 1. From the coefficients on n and M/c it seems that larger households economize on cash holdings, so in our estimated models they will have larger b/c. Also a higher b/c associated with a larger family, gives a lower W/M, as in the table. From the coefficients of $\underline{M}/M$ and on $n/(c/2M)$ it seems that family size does not have a large effect on the precautionary cash holdings.

Table IV shows that similar results are obtained when we use alternative measures of family size (number of family members or of income receivers).

C On the average cash balance M with a precautionary motive

Proposition 1. *Assume that $\pi = 0$ and let λ denote the time elapsed between two consecutive withdrawals. Let $M(\lambda)$ be the average cash balance during this elapsed time, $W(\lambda)$ be the withdrawal at the end of a period* *of length λ and $\underline{M}(\lambda)$ the cash balance just prior to the withdrawal. Let M be the expected value of cash holdings under the invariant distribution and $g(\lambda)$ be the density of the distribution of the lengths. We then have*

$$M(\lambda) \quad = \quad \underline{M}(\lambda) + W(\lambda)/2 = m^* - (c\ \lambda)/2 \tag{1}$$

$$M \quad = \quad \frac{\int_0^\infty M(\lambda)\ \lambda\ g(\lambda)\ \ d\lambda}{\int_0^\infty \lambda\ g(\lambda)\ \ d\lambda} \tag{2}$$

Proof of Proposition 1. Let $t \in [0, \lambda]$ index the time elapsed in an interval of length λ. The law of motion of cash and the optimal policy imply that cash holdings obey $m(t) = m^* - c\lambda$ for $t \in [0, \lambda)$ and $m(\lambda) = m^*$. $W(\lambda) = m^+(\lambda) - m^-(\lambda)$ and $m^* = W(\lambda) + \underline{M}(\lambda)$ imply equation (1). The ergodic theorem implies, using ω to index the sample space,

$$M = \lim_{T\to\infty} (1/T) \int_0^T m(t, \omega)\ dt \quad \text{ in pr.} \tag{3}$$

from which equation (2) can be derived. □

Remark 1. If the distribution of the length λ is concentrated at a single value $\bar{\lambda}$, as in a deterministic model, then $M = M(\bar{\lambda})$. Then

$$M \quad = \quad M(\bar{\lambda}) = \underline{M}(\bar{\lambda}) + W(\bar{\lambda})/2$$

Remark 2 When the distribution of the length λ is not degenerate then

$$M < \int_0^\infty M(\lambda)\ \ g(\lambda)\ \ d\lambda = \int_0^\infty \underline{M}(\lambda)\ \ g(\lambda)\ \ d\lambda + \frac{1}{2}\int_0^\infty W(\lambda)\ \ g(\lambda)\ \ d\lambda$$

where the inequality follows because $M(\lambda)$ is decreasing in λ. Thus M, the duration weighted expected value of $M(\lambda)$, is smaller than the unweighted expected value in the right hand side of the inequality.

D A model with random consumption jumps

The referee suggested a model with a delay between the decision of a withdrawal and the arrival of the cash and with jumps in consumption as an alternative way to obtain a process for cash balances where withdrawals happen at times where cash balances are positive. We empathize completely with the idea. We analyze it thoroughly in the a variation of the model presented below. Nevertheless find it much easier to do it without the delay, in fact it is not even necessary to obtain that cash is positive at time of some of the withdrawals.[5]

[5] With the delay, one has to distinguish between the time at which a withdrawal is decided, and the time at which it is implemented, i.e. when the cash can be spent. In a deterministic model, this distinctions vanishes, since the future is perfectly forecastable. Then the model just adds (with no reason) a lower bound to the cash balances. Hence, we think that what Referee 3 has in mind is a model where the consumption is random. But in this case we think that a fixed delay makes the model not very tractable. For instance, consider the model with jumps in consumption of a fixed size, say z, that happen with a constant Poisson arrival rate, say κ, and a deterministic time lag between the decision and the actual withdrawal. In this case, suppose that there is a large number of jumps in the time between the decision and the actual withdrawal (i.e. the delay). Then non-negative of cash has to be violated. Summarizing, we find the delay on withdrawing a feature that requires a more complex, and less tractable model. Nevertheless, we think we understand the idea of the example, and find it interesting and important. Based on that we consider a variation on the model that incorporates jumps in consumption (of fixed size z, arriving at a Poisson rate κ), but eliminates the delay on withdrawal. We think that this captures the same logic.

D.1 Overview and main findings

We consider a model where consumption has two components: one is deterministic at a constant rate c per unit of time, $-$as in our previous model$-$ and the other is a jump process. We assume that the jump process occurs with probability κ per unit of time, and that when it happens consumption increases by an amount given by the parameter $z > 0$. With this parametrization, expected consumption per year, equals $c + \kappa z$.

The aim of this extension is to explore the implications of an alternative reason for "precautionary" type of behavior. In this model, there are two types of withdrawals, those that occur when m reaches zero, and those that occur at the time of a jump in consumption if $m < z$. The idea is that at times when consumption jumps, if the money balances at hand m are not large enough to pay for the sudden increase in cash consumption, i.e. if $m < z$ then the agent will withdraw cash, even if cash has not reached zero. Otherwise, the nature of the optimal policy is the same, after withdrawal agents set their cash balances to the optimal replenishment level m^*. Hence in this model, the value of z acts in a similar way as the threshold $\underline{m}$ proposed by referee 3.

We show that solving the Bellman equation is more involved than in our previous case, requiring to solve a delay-differential equation, as opposed to an ordinary differential equation. While we present an algorithm to solve for the parameters that fully characterize the Bellman equation, we do not have a simple close form solution for the optimal policy m^* as we did for our benchmark case.

We also give a characterization of the algorithm to solve for M, W, n and $\underline{M}$ as a function of m^* and the rest of the parameters $(\kappa,\ z,\ c)$. We use this characterization to explore the implications of this model for the following statistics of interest: $M/\left(c + z\kappa\right)$, W/M, n, $\underline{M}/M$, and $n/\left[\left(c + z\kappa\right)/\left(2M\right)\right]$. In particular we solve for these statistics by normalizing $c + \kappa z = 365$ and consider different values $c/\left(c + \kappa z\right)$, m^* and κ. We have chosen the values of m^* given κ so that the implied values of cash relative to cash consumption, $M/\left(c + \kappa z\right)$ will be close to the ones for the Italian households in Table 1. We focus on parametrizations where the fraction of cash consumption that is continuous, $c/\left(c + \kappa z\right)$ is at least 1/3. We view this range as the most interesting and realistic case, since higher values imply that most purchases using cash will correspond to jumps and hence will be of large size.[6]

Here is a summary of the findings for this model, illustrated by figures 1-4 (panels of Figure I; see Section D.4 in this appendix, for a detailed explanation of the construction of these figures). Recall that the only difference with the BT model is that at the time of jumps when $m < z$, agents must withdraw. This introduces the following forces: it tends to reduces the number of withdrawals needed to finance cash consumption for a given cash consumption, since a non-negligible fraction of consumption happens at the time of the withdrawal (see Figures 1 and 4); it tends to increase the size of withdrawals, since the withdrawals at the time of a jump include z (see Figure 2) and it produces a positive average cash balances at the time of withdrawal, since some withdrawals happen when $m < z$ (see Figure 4). Thus, the number of withdrawals, n, that correspond to cash holding relative to cash purchases, $M/\left(c + z\kappa\right)$, are close to or smaller than the ones in the BT model (see Figure 1). This result goes against most of the data, especially for those with ATM cards (notice how low is the number of withdrawals in the model relative to those in Table 1 in the paper). This can also be seen in the statistic $n/\ \left[\left(c + \kappa z\right)/\left(2M\right)\right]$, the ratio of the actual number of withdrawals to the implied number of withdrawals from the BT model (this statistics is one in the BT model). This is closer of smaller than one for this model, while it is much bigger in our data for Italian households (see Table 1 in the paper). Finally, one feature of this model that agrees with the data is that it easily produces positive and large values of $\underline{M}/M$, see Figure 4.

Summarizing, on the positive side this model allows to have observations with $W/M \geq 2$ or equivalently with values of $M <\ \left(c + \kappa z\right)/2n$, as we observe for a small number of households, especially those without ATM cards. On the negative side, we find the model less tractable than our benchmark model, and inconsistent with the large number of withdrawals that characterize the behavior of the typical household. Furthermore, we do not have time series data (at high frequency) on cash consumption to retrieve infor-

[6] For example, if $\kappa z = 2/3$ of total annual cash consumption, and cash consumption is about 2/3 of total consumption, for $\kappa = 10$, then the purchases z will be about 4.5 % of annual total consumption. For completeness, we discuss the other cases in detail in the last section of this document.

mation on z and κ. Hence, we think that the jumps in consumption are distinguishable from the model with random free withdrawals only in terms of some of their implications. We prefer our benchmark model (the one with random free withdrawals) because it is simpler, it allows a more thorough understanding, characterization, and comparative statistics; it introduces fewer parameters (p versus z and κ); and it is roughly consistent with more patterns of the data.

D.2 The Bellman Equation in the model with consumption jumps

The Bellman equation for $m > 0$ becomes:

$$\begin{aligned} rV(m) &= Rm + p\left[\min_{\hat{m}} V(\hat{m}) - V(m)\right] \\ &+\kappa \min\left[b + \min_{\hat{m}} V(\hat{m}) - V(m)\,,\ V(m-z) - V(m)\right] \\ &+V'(m)(-c - \pi m) \end{aligned}$$

The term $\min\left[b + \min_{\hat{m}} V(\hat{m}) - V(m)\,,\ V(m-z) - V(m)\right]$ takes into account that after the jump in consumption the agent can decide to withdraw cash, or otherwise her cash balances becomes $m - z$. As before we let m^* solve: $m^* = \arg\min_{\hat{m}} V(\hat{m})$ we also let $V^* \equiv V(m^*)$. Non-negativity of cash gives

$$V(m) = V^* + b \text{ for } m \leq 0.$$

We look for a solution of the form of an Ordinary Differential Equation (ODE):

$$(r + p + \kappa) V(m) = Rm + (p + \kappa) V^* + \kappa b + V'(m)(-c - \pi m)$$

for all $0 \leq m \leq z$ and of a Delay-Differential Equation (DDE):

$$(r + p + \kappa) V(m) = Rm + pV^* + \kappa V(m - z) + V'(m)(-c - \pi m)$$

for $z \leq m \leq m^{**}$.

D.2.1 Characterization of the Bellman Equation

We can further characterize the solution by splitting the domain of V, $[0, m^*]$ into J intervals. The first $J-1$ intervals are of width z and are given by $[jz, (j+1)z]$ for $j = 0, 1, ..., J-1$. The last interval is given by $[Jz,\ \min\{m^*,\ (J+1)z\}]$. We index the solution of the ODE in each interval by j as follows:

$$(r + p + \kappa) V_0(m) = Rm + (p + \kappa) V^* + \kappa b + V_0'(m)(-c - \pi m)$$

for $0 \leq m \leq z$, V_0 and the DDE:

$$(r + p + \kappa) V_j(m) = Rm + pV^* + \kappa V_{j-1}(m - z) + V_j'(m)(-c - \pi m)$$

for $j\ z \leq m \leq \min\{(j+1)\ z, m^*\}$ and $1 \leq j \leq J$.

Notice that given, $V^* \geq 0$ and $V(0) = b + V^*$ one can readily solve the ODE for V_0 as done in the model with $\kappa = 0$. For the other intervals, given V^* we can solve V_j recursively, since in the j segment we use the solution for the $j-1$ segment, i.e. we can treat the DDE as an ODE. Since V is continuous, we require that

$$V_j(zj) = V_{j-1}(zj)$$

for $j = 1, 2, ..., J-1$. Notice that this equality also implies that the derivatives of V_j and V_{j-1} agree at this point for $j \geq 2$. Finally we look for the J segment for which:

$$V'_{J-1}(m^*) = 0 \text{ and } V_{J-1}(m^*) = V^* \; .$$

This characterization immediately implies an algorithm to solve for the solution: Guess for a value of V^* solve for $V_0, ..., V_{J-1}$, where J is found so that there is a point m^* for which $V'_{J-1}(m^*) = 0$. Finally, check if $V_{J-1}(m^*) = V^*$.

D.2.2 Solution of the Bellman equation for the $\pi = 0$ case

The following proposition gives the functional form for the solution of the Bellman equation, and display the equations on the constants that give the solution for the ODE-DDE for the Bellman equation.

Proposition 2. *a) Given* (V^*, m^*) *let* J *be the smallest integer such that* $Jz \geq m^*$. *The functions* $V_j : [zj,\ z(j+1)] \to R$ *for* $j = 0, ..., J-1$:

$$V_j(m) = A_j + D_j(m - z\ j) + \exp(\lambda(m - zj)) \sum_{i=0}^{j} B_{j,i}\ (m - z\ j)^i$$

where the constants $\lambda,\ A_j, D_j$ *and* $B_{j,i}$ *satisfy:*

$$\begin{aligned} \lambda &= \frac{r + p + \kappa}{-c}, \\ D_0 &= \frac{R}{(r + p + \kappa)}, \\ A_0 + B_{0,0} &= b + V^* \end{aligned}$$

and for $j = 0, 1, ..., J-2$:

$$\begin{aligned} D_{j+1} &= (-\ 1\ /\ \lambda)\left[\frac{R}{c} + \frac{\kappa}{c}D_j\right], \\ A_{j+1} &= (1/\lambda)\left(D_{j+1} - \frac{pV^*}{c} - \frac{\kappa}{c}\left[A_j - D_j\ z(j+1)\right]\right) + D_{j+1}z(j+1), \\ B_{j+1,0} &= A_j + D_j z + \ \exp(\lambda z) \sum_{i=0}^{j} B_{j,i}\ (z)^i - A_{j+1},\ \textit{and} \end{aligned}$$

$$B_{j+1,i+1} = \frac{1}{i+1}\frac{\kappa}{c}B_{j,i}$$

for $i = 0,\ 1, 2, ..., j$.

b) Given $\{A_j, D_j,\ B_{ji}\}$, $(V^*,\ m^*)$ *must satisfy:*

$$\begin{aligned} V^* &= A_{J-1} + D_{J-1}(m^* - z\ (J-1)) + \exp(\lambda(m^* - z(J-1))) \sum_{i=0}^{J-1} B_{J-1,i}\ (m^* - z\ (J-1))^i, \\ 0 &= D_{J-1} + \exp(\lambda(m^* - z(J-1))) \left[\sum_{i=0}^{J-1} B_{J-1,i}\ \left(\lambda(m^* - z\ (J-1))^i + i\ (m^* - z\ (J-1))^{i-1}\right)\right] \end{aligned}$$

D.3 Characterization of cash management statistics

We can determine M, W, $\underline{M}$ and n by finding the invariant distribution h the expected number of withdrawals, n and using them to compute the remaining statistics (M,$\underline{M}$ and W), as was done in the model with random free withdrawals. We do so in the appendix, but the derivation and calculations of n and h are much more involved than the one for the model with random free withdrawals. Instead here we derive then in a different manner, without solving for h, directly finding an expression for the statistic of interest. We define M w,$\underline{M}$ and n as the expected discounted integral of the respective quantities, and then take the discount rate to zero.[7]. In particular let:

$$
\begin{aligned}
M(m) &= E\left[\rho \int_0^{\infty} e^{-\rho} m(t)\, dt \mid m\right] \\
w(m) &= E\left[\rho \sum_{j=0}^{\infty} e^{-\rho \tau_j} w(\tau_j)\ dt \mid m\right] \\
\underline{m}(m) &= E\left[\rho \sum_{j=0}^{\infty} e^{-\rho \tau_j} m\left(\tau_j^-\right)\ dt \mid m\right] \\
n(m) &= E\left[\rho \sum_{j=0}^{\infty} e^{-\rho \tau_j}\ dt \mid m\right]
\end{aligned}
$$

where τ_j are the times at which a withdrawal happens, and where $w(\tau_j)$ is the size of the corresponding withdrawal. We are interested in

$$M = \lim_{\rho \to 0} M(m),\ w = \lim_{\rho \to 0} w(m),\ \underline{m} = \lim_{\rho \to 0} \underline{m}(m),\ n = \lim_{\rho \to 0} n(m).$$

Note that w is the expected value of the total amount of withdrawals during a period, and hence average withdrawal size is

$$W = \frac{w}{n}.$$

Likewise, $\underline{m}$ is the expected value of the total amount of cash at the time of a withdrawal, and hence the average cash at the time of a withdrawal is

$$\underline{M} = \frac{\underline{m}}{n}.$$

These functions satisfy the following Bellman equations, which, in order to simplify the solution, we only write for the case of $\pi = 0$. The logic for them is the same as the one for the value function, so we present directly their characterization in segments $[zj,\ z(j+1)]$. For $m \in [0, z]$

$$
\begin{aligned}
(\rho + \kappa + p) M_0(m) &= \rho\, m - M_0'(m)\, c + (\kappa + p) M^* \\
(\rho + \kappa + p) w_0(m) &= \rho (\kappa + p)(m^* - m) + \rho \kappa z - w_0'(m)\, c + (\kappa + p) w^* \\
(\rho + \kappa + p) \underline{m}_0(m) &= \rho (\kappa + p) m - \underline{m}_0'(m)\, c + (\kappa + p) \underline{m}_0^* \\
(\rho + \kappa + p) n_0(m) &= \rho (\kappa + p) - n_0'(m)\, c + (\kappa + p) n^*
\end{aligned}
$$

which can be written as:

$$(\rho + \kappa + p) F_0(m) = \rho\ \sigma_0\ m + \rho\ \alpha_0 - F_0'(m)\, c + (\kappa + p) F^*$$

for suitable choices of the constants σ_0 and α_0 (see the appendix for details)

[7]We implement the limit numerically, by solving it for a small value of ρ

For $m \in [zj,\ z(j+1)]$ for $j = 1, 2, ...,\ J-2$

$$\begin{aligned}
(\rho+\kappa+p)\, M_j(m) &= \rho\, m - M_j'(m)\, c + \kappa M_{j-1}(m-z) + pM^* \\
(\rho+\kappa+p)\, w_j(m) &= \rho p\,(m^*-m) - w_j'(m)\, c + \kappa\ w_{j-1}(m-z) + p\ w^* \\
(\rho+\kappa+p)\, \underline{m}_j(m) &= \rho\ pm - \underline{m}_j'(m)\, c + \kappa\ \underline{m}_{j-1}(m-z) + p\underline{m}^* \\
(\rho+\kappa+p)\, n_j(m) &= \rho p - n_j'(m)\, c + \kappa n_{j-1}(m-z) + pn^*
\end{aligned}$$

which is written as

$$(\rho+\kappa+p)\, F_j(m) = \rho\ \sigma\ m + \rho\alpha - F_j'(m)\, c + \kappa F_{j-1}(m-z) + pF^*$$

for some suitable choices of α and σ (see the appendix for details). Continuity of these function across the segments gives:

$$\begin{aligned}
M_j(zj) &= M_{j-1}(zj), \\
w_j(zj) &= w_{j-1}(zj), \\
\underline{m}_j(zj) &= \underline{M}_{j-1}(zj), \\
n_j(zj) &= n_{j-1}(zj)
\end{aligned}$$

and in general

$$F_j(zj) = F_{j-1}(zj)$$

for $j = 1, 2, ...,\ J-1$ and the appropriate boundary codnditions at $m = 0$:

$$\begin{aligned}
M_0(0) &= M^*, \\
w_0(0) &= \rho\ m^* + w^*, \\
\underline{m}_0(0) &= \underline{m}^*, \\
n_0(0) &= \rho + n^*
\end{aligned}$$

or in general:

$$F_0(0) = \rho\ \alpha^* + F^*$$

for a suitable choice of α^* (see the appendix for details).

Now we can write the solution for F as a function of m^*, σ, σ_0, α, α_0 and α^*.

Proposition 3. *The ODE-DDE for F has the following solution. Let m^*, σ ,σ_0, α , α_0 and α^* be given. Using m^* define J as the smallest integer so that $Jz \geq m^*$. Then $F_j : [zj,\ z(j+1)] \to R$ has the form*

$$F_j(m) = G_j + S_j(m-zj) + \sum_{i=0}^{j} H_{ji} \exp(\lambda(m-zj))\,(m-zj)^i$$

for some constants G_j, S_j, H_{ij} and λ. These constants solve:

$$\begin{aligned}
\lambda &= \frac{\rho+\kappa+p}{-c}\ , \\
S_0 &= \frac{\rho}{\rho+\kappa+p}\sigma_0\ , \\
G_0 &= \frac{\rho}{\rho+\kappa+p}\ \alpha_0 - \frac{c}{\rho+\kappa+p}S_0 + \frac{(\kappa+p)}{\rho+\kappa+p}F^*, \\
H_{00} &= \rho\alpha^* + F^* - G_0\ .
\end{aligned}$$

for $j = 0, 1, 2, ..., J-2$:

$$\begin{aligned} S_{j+1} &= \frac{\rho}{\rho+\kappa+p}\ \sigma\ + \frac{\kappa}{\rho+\kappa+p} S_j\ , \\ G_{j+1} &= \frac{\rho\alpha + pF^*}{\rho+\kappa+p} + \frac{\kappa}{\rho+\kappa+p}\left[G_j - S_j z\left(j+1\right)\right] - \frac{c}{\rho+\kappa+p} S_{j+1} + S_{j+1} z\left(j+1\right), \\ H_{j+1,0} &= G_j + S_j\ z + \sum_{i=0}^{j} H_{ji} \exp\left(\lambda z\right)\left(z\right)^i - G_{j+1} \end{aligned}$$

and

$$H_{j+1,\ i} = \frac{1}{i}\frac{\kappa}{c}\ H_{j,i-1}$$

for $i = 1, 2, ..., \ j+1$.

Finally, F^* *must solve:*

$$F^* = G_{J-1} + S_{J-1}\left(m^* - z\left(J-1\right)\right) + \sum_{i=0}^{J-1} H_{J-1\ i} \exp\left(\lambda\left(m^* - z\left(J-1\right)\right)\right)\ \left(m^* - z\left(J-1\right)\right)^i$$

D.4 Code and Figures

The set of figures below use the matlab code cons_jump_formulas.m. That code uses the expressions above to solve for M/c, W/M, n, $\underline{M}/M$ and $n/\left[\left(c+\kappa z\right)/\left(2M\right)\right]$ for several combination of the jump intensity κ, composition of cash consumption $c/\left(c+\kappa z\right)$, and replenishing cash balance threshold m^*. We plot the resulting values in different graphs. Using the homogeneity of the model, we normalize expected cash consumption per year, $c + \kappa z = 365$ so that $m^* = 20$ is interpreted as a stock of cash that is equivalent to 20 days of expected cash consumption. In each graph we display 9 different solid lines, each line correspond to a combination of (κ, m^*). The three values of κ are 5, 10 and 20 expected jumps per year; the 3 values for m^* are 20, 30 and 40 cash holding, in daily expected cash consumption units. The different values of κ are denoted by lines of different colors, and the different values of m^* are indicated by labels at the end of each line. The points of a line correspond to different values of the share of cash purchases accounted for the continuous consumption, $c/\left(c+z\kappa\right)$ going from $z = 0$ and hence $c = 365$ up to $\bar{z}$ chosen to that $\bar{z}\kappa\ /\ 365 = 2/3$ i.e. so that cash consumption that occurs in jumps account for up to 2/3 of cash consumption. The figures include also a dotted line with the prediction corresponding to the BT model. For completeness we also considered the case where $2/3 < \bar{z}\kappa \leq 365$ and $c = 0$, even though we view such a high value of $z\kappa$ as unrealistic (it is implausible that unexpected cash transaction would be so large, as discussed in the introduction and footnote 6 therein).

The first lines of the code cons_jump_formulas.m allows to change the values of κ, m^*, p and $\bar{z}$ for the 9 lines.

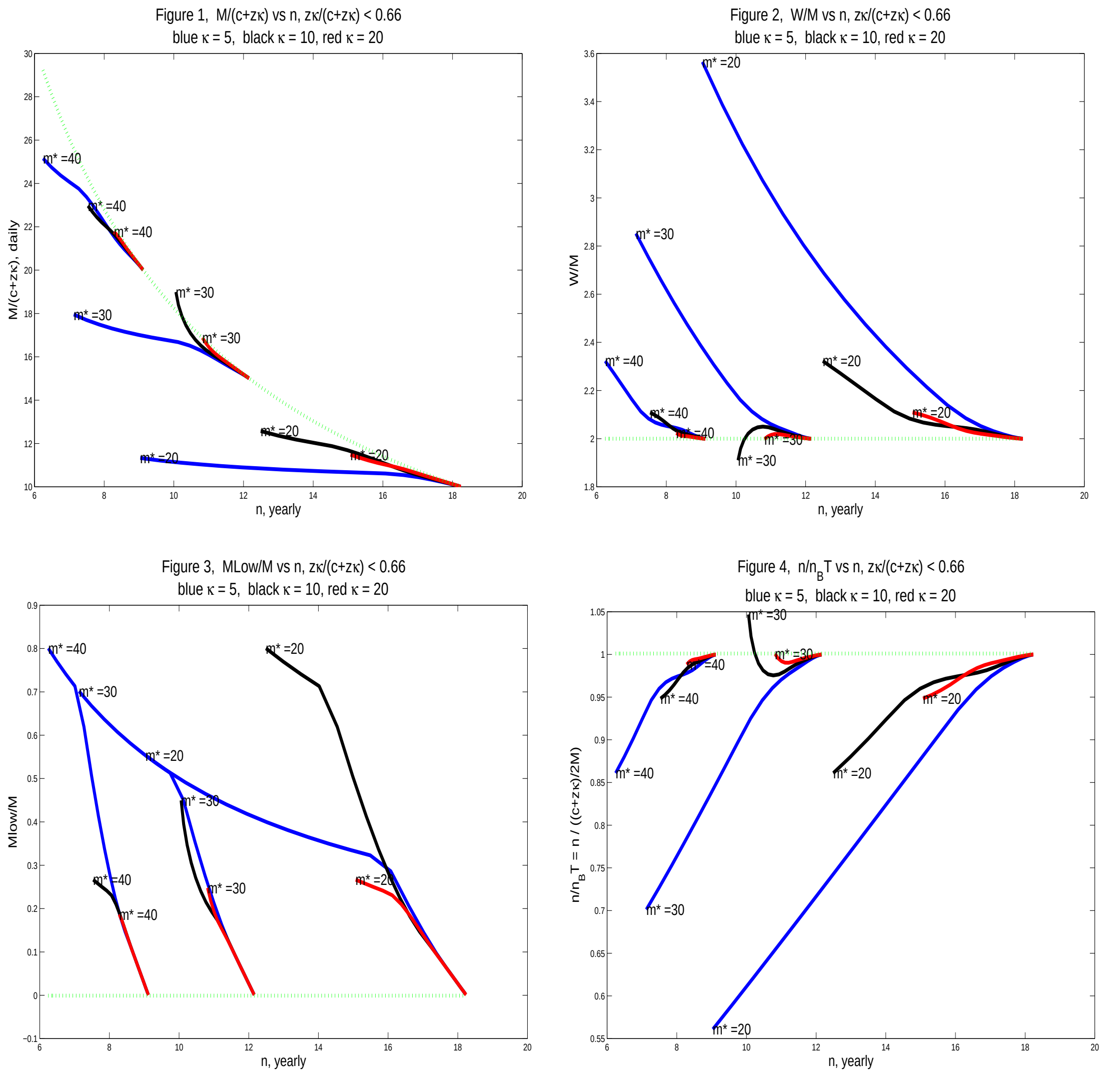
I. Case: $0 \leq z\kappa/365 \leq 2/3$.

This case is obtained by running the matlab code cons_jump_formulas.m setting high_zkappa = 2/3*365 and setting Fign_intial=1.

Figures 1, 2, 3 and 4 plot curves with n the expected number of withdrawals per year in the horizontal axis, and different statistics in the vertical axis. Figure 1 plots in the vertical axis the ratio of average cash balances to expected daily cash consumption $M/\left(c+z\kappa\right)$ Figure 2 plots average withdrawal size relative to average cash balances, W/M, Figure 3 plots average cash at the time to withdrawal to average cash holdings, $\underline{M}/M$ and Figure 4 plots the ratio of the average number of withdrawals to the implied number in the BT model, $n\left[\left(c+\kappa z\right)/\left(2M\ \right)\right]$.

It is instructive to analyze the cases that correspond to the extremes of each of the curves. Consider the extreme that correspond to $c = 365$ and $z = 0$. In all the figures, the 3 curves for the same value of m^*, all start from the same point. The point is the same for all the values of κ because with $z = 0$ the value κ is immaterial. For instance that Figure 1, all this values $W/M = 2$ in Figure 2 they trace the $2M$ $n = 365$, in Figure 3 the correspond to the value of $n/[365/2M] = 1$. Notice that the point of this extreme coincide with the BT model, whose predictions are in each figure recorded by the dotted line. The other extreme correspond to the value of the ratio of purchases accounted by the jumps, $\bar{z}\kappa/365$ equal to $2/3$.

Figure I: Predictions of model with consumption jumps ($z\kappa/365 \leq 2/3$)



From Figures 1, 2 and 4 (the Panels of Figure I) it is clear that the statistics of this model are on the opposite side of the BT model, relative to the data: the model predicts fewer withdrawals, each of a larger dimension. In particular, in Figure 1 they are below the BT line, in Figure 2 they are above the BT line, and in Figure 4 they are below the BT line. It is also clear that this model has no problem producing large values of $\underline{M}/M$ (see Figure 3).

II. Case: $0 \leq z\kappa/365 \leq 1$.

This case is obtained by running the matlab code cons_jump_formulas.m setting high_zkappa = 0.99*365 and setting Fign_intial=5.

Now we analyze the case where we allow higher values of z, namely we continue the curves up to the point where $\bar{z}\ \kappa/\ 365 = 1$. The extreme of these curves require some discussion. Depending on the value of κ and m^*, on this extreme of the curve, z can be even bigger than m^*. We consider first the case where $z > m^*$, or equivalently, when $365 > m^*\ \kappa$. In this case the jump in consumption z is so large than after the first jump there is a withdrawal. Hence, there are as many withdrawals as jumps, so the average number of withdrawals is $n = \kappa$ (In the figures this corresponds to $\kappa = 5,\ 10$ and 20). Since $c = 0$, cash holdings are constant at m^* before the withdrawal, and equal to this amount at the time of the withdrawal, hence $M = m^*$ and $\underline{M} = m^* = M$. Finally, the withdrawal size is $W = z > m^*$, and thus $W/M = z/m^* = 365\ /\ (\kappa\ m^*)$. Summarizing:

$$\begin{aligned} n &= \kappa, \\ \frac{W}{M} &= 365\ /\ (\kappa\ m^*)\,, \\ \frac{\underline{M}}{M} &= 1. \end{aligned}$$

Now consider the general case, including the one for which $z < m^*$,or equivalently $365 < m^*\ \kappa$. In this case J jumps are required for a withdrawal where J is the smallest integer such that $zJ \geq m^*$. In this case a withdrawal happens after the J jump, and hence $n = \kappa/J$. Cash holdings start at m^*. stay constant until the next jump, at which time they decrease by z, until the Jth jump where there is a withdrawal. Hence:

$$\begin{aligned} M &= \frac{1}{J}m^* \ + \frac{1}{J}\left(m^* - z\right) + \frac{1}{J}\left(m^* - z\right) + \cdots + \frac{1}{J}\left(m^* - \left(J-1\right)z\right) \\ &= m^* - \frac{(J-1)}{2}z \end{aligned}$$

After the Jth jump, a withdrawal takes place and hence W :

$$W = m^* + z - \left(\left(m^* - \left(J-1\right)z\right)\right) = Jz.$$

Thus

$$\frac{W}{M} = \frac{Jz}{m^* - \frac{(J-1)}{2}z}$$

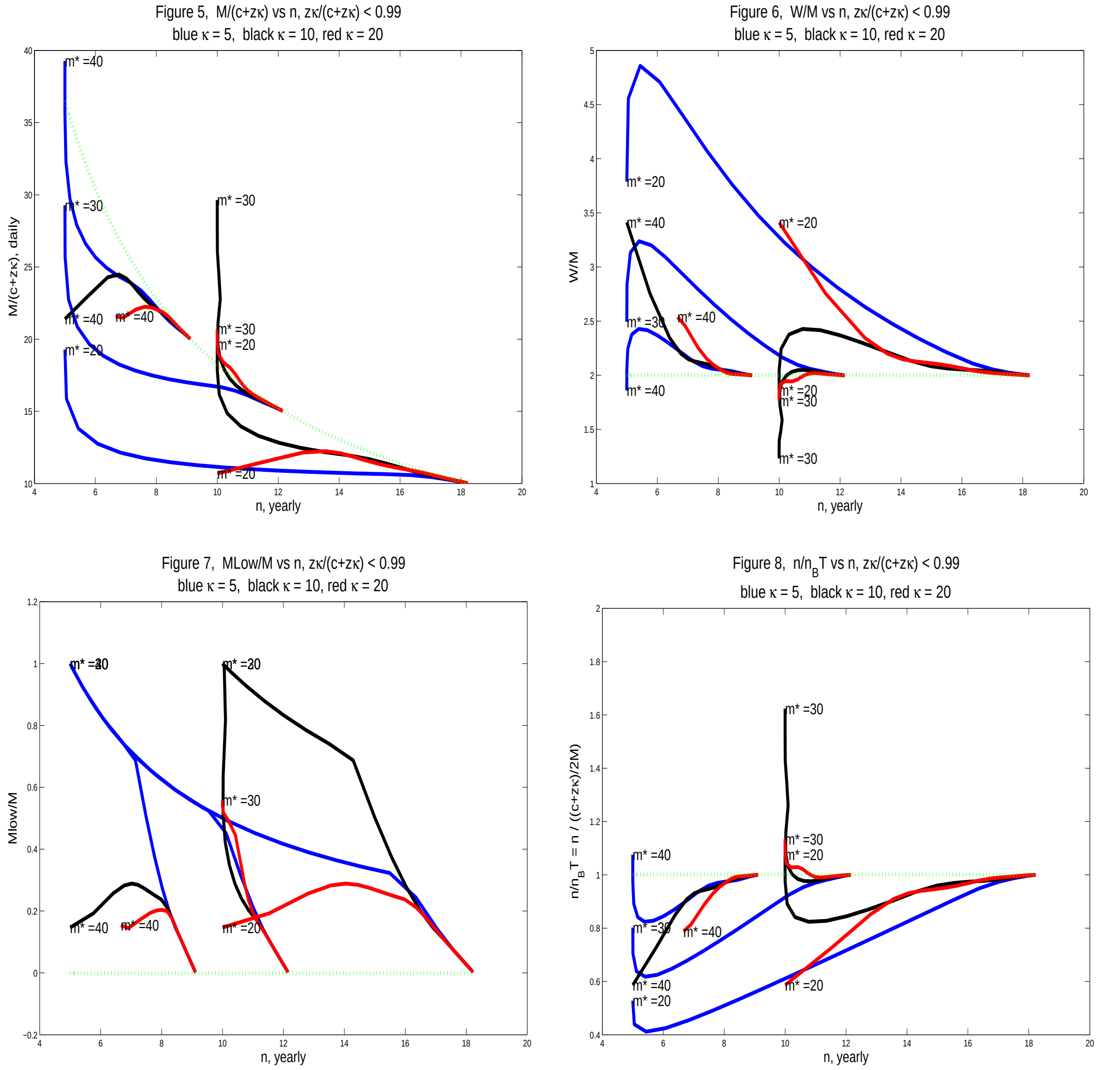
After $J-1$ withdrawals, cash balances are $m^* - \left(J-1\right)z$ This is the amount that agents will have at the time of the withdrawal, hence

$$\frac{\underline{M}}{M} = \frac{m^* - \left(J-1\right)z}{m^* - \frac{(J-1)}{2}z}$$

These formulas are complicated by the fact that m^* may not be an exact multiple of z. To simplify matters, assume that it is, so that $Jz = m^*$ then or

$$\begin{aligned} n &= \frac{\kappa}{J} = \frac{365}{m^*} = \kappa\ \frac{1}{\kappa m^*/365} \\ \frac{\underline{M}}{M} &= \frac{m^*/z - (J-1)}{m^*/z - \frac{(J-1)}{2}} = \frac{2}{\frac{\kappa m^*}{365} + 1} < 1, \\ \frac{W}{M} &= \frac{Jz}{m^* - \frac{(J-1)}{2}z} = \frac{2}{1 + \frac{365}{m^*\kappa}} < 2 \end{aligned}$$

Figure II: Predictions of model with consumption jumps ($z\kappa/365 \leq 1$)



Figures 5 to 8 (panels of Figure II) plot the same variables that are plotted in as Figure Figure I, except that the curves are continued for values of z until $z\kappa = 365$. The implications of these curves vis-a-vis the data are similar to the ones of Figures discussed above. The only notable exception is say for $m^* = 30$ and $\kappa = 10$. In this case the model, for values of $z\kappa$ close to 365 can generate values of $n/\left[(c + z\kappa)/(2M)\right]$ substantially larger than one, and values of W/M substantially smaller than 2. Nevertheless this correspond to values of $\underline{M}/M$ very close to one, and to values of n close to 10 which are both counterfactual when compared with the statistics in Table 1 of the paper. Indeed these cases correspond exactly to the analytical formulas presented above with $m^* \leq z$ and $z\kappa = 365$.

E A model with costly random withdrawals

The dynamic model discussed in the paper has the unrealistic feature that agents withdraw every time a match with a financial intermediary occurs, thus making as many withdrawals as contacts with the financial intermediary, many of which of a very small size. In this section we extend the model to the case where the withdrawals (deposits) done upon the random contacts with the financial intermediary are subject to a fixed cost f, assuming $0 < f < b$. The model produces a more realistic depiction of the distribution of withdrawals, by limiting the minimum withdrawal size. In particular, we show that the minimum withdrawal size is determined by the fixed cost relative to the interest cost, i.e. f/R, and that it is independent of p. On the other hand, if f is large relative to b, the predictions gets closer to the ones of the BT model. Indeed, as f goes to b, then there is no advantage of a chance meeting with the intermediary, and hence the model is identical to the one of the previous section, but with $p = 0$.

In this section we formulate the dynamic programming problem for $f > 0$, solve its Bellman equation and characterize its optimal decision rule. We also derive the corresponding invariant distribution and the expressions for n, M, W, $\underline{M}$. As several features of this case are similar to the previous one we streamline the presentation and do not report results on comparative statics or welfare.

We skip the formulation of the total cost problem, that is exactly parallel to the one for the case of $f = 0$. Using notation that is analogous to the one that was used above, the Bellman equation for this problem when the agent is not matched with a financial intermediary is given by:

$$rV(m) = Rm + p \min \{V^* + f - V(m), 0\} + V'(m)(-c - m\pi) \tag{4}$$

where $V^* \equiv \min_{\hat{m}} V(\hat{m})$ and $\min\{V^* + f - V(m), 0\}$ takes into account that it may not be optimal to withdraw/deposit for all contacts with a financial intermediary. Indeed, whether the agent chooses to do so will depend on her level of cash balances.

We will guess, and later verify, a shape for $V(\cdot)$ that implies a simple threshold rule for the optimal policy. Our guess is that $V(\cdot)$ is strictly decreasing at $m = 0$ and single peaked attaining a minimum at a finite value of m. Then we guess that there will be two thresholds, $\underline{m}$ and $\bar{m}$, that satisfy:

$$V^* + f = V(\underline{m}) = V(\bar{m}) \quad . \tag{5}$$

Thus solving the Bellman equation is equivalent to finding 5 numbers m^*, m^{**}, $\underline{m}$, $\bar{m}$, V^* and a function $V(\cdot)$ such that:

$$V^* = V(m^*) \ , \ 0 = V'(m^*) \tag{6}$$

$$V(m) = \begin{cases} \dfrac{Rm + p(V^* + f) - V'(m)(c + m\pi)}{r + p} & \text{if } m \in (0, \ \underline{m}) \\ \dfrac{Rm - V'(m)(c + m\pi)}{r} & \text{if } m \in (\underline{m}, \bar{m}) \\ \dfrac{Rm + p(V^* + f) - V'(m)(c + m\pi)}{r + p} & \text{if } m \in (\bar{m}, \ m^{**}) \end{cases} \tag{7}$$

and the boundary conditions:

$$V(0) = V^* + b \ , \ V(m) = V^* + b \text{ for } m > m^{**} \quad . \tag{8}$$

Hence the optimal policy in this model is to pay the fixed cost f and withdraw cash if the contact with the financial intermediary occurs when cash balances are in $(0, \underline{m})$ range, or to deposit if cash balances are larger than $\bar{m}$. In either case the withdrawal or deposits is such that the post transfer cash balances are equal to m^*. If the agent contacts a financial intermediary when her cash balances are in $(\underline{m}, \ \bar{m})$ then, no action is taken. If the agent cash balances get to zero, then the fixed cost b is paid and after the withdrawal the cash balances are set to m^*. Notice that $m^* \in (\underline{m}, \ \bar{m})$. Hence in this model withdrawals have a minimum size given by $m^* - \underline{m}$. This is a more realistic depiction of actual cash management.

Now we turn to the characterization and solution of the Bellman equation.

Proposition 4. *For a given $V^*, \underline{m}, \bar{m}, m^{**}$ satisfying $0 < \underline{m} < \bar{m} < m^{**}$:*
The solution of (7) for $m \in (\underline{m}, \bar{m})$ is given by:

$$\begin{aligned} V(m) &= \varphi(m, A_\varphi) \equiv \\ &\equiv \frac{-Rc/(r+\pi)}{r} + \frac{R\, m}{r+\pi} + \left(\frac{c}{r}\right)^2 A_\varphi \left[1 + \pi \frac{m}{c}\right]^{-\frac{r}{\pi}} \end{aligned} \tag{9}$$

for an arbitrary constant A_φ.

Likewise, the solution of (7) for $m \in (0,\ \underline{m})$ or $m \in (\bar{m},\ m^{**})$ is given by:

$$\begin{aligned} V(m) &= \eta(m, V^*, A_\eta) \equiv \\ &\equiv \frac{p(V^*+f) - \frac{R\ c}{r+p+\pi}}{r+p} + \frac{R\ m}{r+p+\pi} + \left(\frac{c}{r+p}\right)^2 A_\eta \left[1 + \pi \frac{m}{c}\right]^{-\frac{r+p}{\pi}} \end{aligned} \tag{10}$$

for an arbitrary constant A_η.

Proof. The proposition is readily verified by differentiating (9) and (10) in their respective domains. □

Next we are going to list a system of 5 equations in 5 unknowns that describes a C^1 solution of $V(m)$ on the range $[0, m^*]$. The unknowns in the system are $V^*, A_\eta, A_\varphi, \underline{m}, m^*$. Using Proposition 4, and the boundary conditions (5), (6) and (8), the system is given by the following 5 equations:

$$\varphi_m(m^*, A_\varphi) = 0 \tag{11}$$

$$\varphi(m^*, A_\varphi) = V^* \tag{12}$$

$$\eta(\underline{m}, V^*, A_\eta) = V^* + f \tag{13}$$

$$\eta(0, V^*, A_\eta) = V^* + b \tag{14}$$

$$\varphi(\underline{m}, A_\varphi) = V^* + f \quad . \tag{15}$$

In the proof of Proposition 5 we show that the solution of this system can be found by solving one non-linear equation in one unknown, namely $\underline{m}$. Once the system is solved it is straightforward to extend the solution to the range: (m^*, ∞).

Proposition 5. *There is a solution for the system (11)-(15). The solution characterizes a C^1 function that is strictly decreasing on $(0, m^*)$, convex on $(0, \bar{m})$ and strictly increasing on (m^*, m^{**}). This function solves the Bellman equations described above. The value function satisfies*

$$V(0) = \frac{R}{r}\ m^* + b \quad . \tag{16}$$

Proof. See Appendix E.1.

Next we present a proposition about the determinants of the range of inaction $m^* - \underline{m}$, or equivalently the size of the minimum withdrawal.

Proposition 6. *The scaled range of inaction* $(m^* - \underline{m}) / (c + m^*\pi)$ *solves*

$$\frac{f}{R(c+m^*\pi)} = \left(\frac{m^*-\underline{m}}{c+m^*\pi}\right)^2 \left[\frac{1}{2} + \sum_{k=1} \frac{1}{(k+2)!} \left(\frac{m^*-\underline{m}}{c+m^*\pi}\right)^k \Pi_{j=2}^{k+1}(r+j\pi)\right], \tag{17}$$

hence it can be written as

$$\frac{m^*-\underline{m}}{c+m^*\pi} = \sqrt{\frac{2\ f}{R\ \ (c+\pi m^*)}} + o\left(\left(\frac{f}{R\ \ (c+\pi m^*)}\right)^2\right), \tag{18}$$

and for $\pi = 0$ *it is increasing in* f/R *with elasticity smaller than* $1/2$.
Proof. See Appendix E.1.

The quantity $c + m^*\pi$ is a measure of the use of cash per period when $m = m^*$. The quantity $m^* - \underline{m}$ also measures the size of the smallest withdrawal. Hence $(m^* - \underline{m}) / (c + m^*\pi)$ is a normalized measure of the minimum withdrawal. The proposition shows that for $\pi = 0$ the minimum withdrawal does not depend on p and b, and that, as the approximation above makes clear, it is analogous to the withdrawal of the BT model facing a fixed cost f and an interest rate R. Quantitatively, these properties continue to hold for $\pi > 0$.

The next proposition examines the expected number of withdrawals n.

Proposition 7. *The expected number of cash withdrawals per unit of time,* $n\,(m^*/c,\ \underline{m}/c, \pi, p)$, *is*

$$n = \frac{p}{(p/\pi)\log(1+(m^*-\underline{m})\,\pi/c) + 1 - (1+\underline{m}\pi/c)^{-\frac{p}{\pi}}} \tag{19}$$

and the fraction of agents with cash balances below $\underline{m}$ *is given by*

$$H(\underline{m}) = \frac{1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}{(p/\pi)\log(1+(m^*-\underline{m})\,\pi/c) + 1 - (1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}\quad . \tag{20}$$

Proof. See Appendix E.1.

Inspection of equation (19) confirms that when $m^* > \underline{m}$ the expected number of withdrawals (n) is no longer bounded below by p. Indeed, as $p \to \infty$ then $n \to [(1/\pi)\log(1+(m^*-\underline{m})\,\pi/c)]^{-1}$, which is the reciprocal of the time that it takes for an agent that starts with money holding m^* (and consuming at rate c when the inflation rate is π) to reach real money holdings $\underline{m}$.

As in the case of $f = 0$, for any $m \in [0, \underline{m}]$ the density $h(m)$ solves the ODE given by equation

$$\frac{\partial h(m)}{\partial m} = \frac{(p-\pi)}{(\pi m + c)} h(m) \tag{21}$$

The reason is that in this interval the behavior of the system is the same as the one for $f = 0$. On the interval $m \in [\underline{m}, m^*]$ the density $h(m)$ solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{-\pi}{(\pi m + c)} h(m)\quad . \tag{22}$$

In this interval the chance meetings with the intermediary do not trigger a withdrawal, hence it is as if $p = 0$.

Proposition 8. *For $H(\underline{m})$ as given in (20), the CDF $H(m)$ for $m \in [0, \underline{m}]$ is*

$$H(m) \quad = \quad H(\underline{m}) \frac{\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} - 1}{\left(1 + \frac{\pi}{c} \underline{m}\right)^{\frac{p}{\pi}} - 1} \tag{23}$$

for $m \in [\underline{m}, m^*]$

$$H(m) \quad = \quad [1 - H(\underline{m})] \frac{\log\left(1 + \frac{\pi}{c} m\right) - \log\left(1 + \frac{\pi}{c} m^*\right)}{\log\left(1 + \frac{\pi}{c} m^*\right) - \log\left(1 + \frac{\pi}{c} \underline{m}\right)} + 1 \quad . \tag{24}$$

Proof. See Appendix E.1.

Using the previous density, the average money holdings $M\left(\frac{m^*}{c}, \frac{\underline{m}}{c}, \pi, p\right)$ is

$$M = \int_0^{\underline{m}} m h(m)\, dm + \int_{\underline{m}}^{m^*} m h(m)\, dm$$

whose closed form expression can be found in the online Appendix L.5.

The average withdrawal $W\left(\frac{m^*}{c}, \frac{\underline{m}}{c}, \pi, p\right)$ is given by

$$W = m^* \left[1 - \frac{p}{n} H(\underline{m})\right] + \left[\frac{p}{n} H(\underline{m})\right] \frac{\int_0^{\underline{m}} (m^* - m)\, h(m)\, dm}{H(\underline{m})} \tag{25}$$

whose closed form expression can be found in the online Appendix L.6. To understand this expression notice that $n - pH(\underline{m})$ is the number of withdrawals in a unit of time that occur because agents reach zero balances, so if we divide it by the total number of withdrawals per unit of time, n, we obtain the fraction of withdrawals that occur when agents reach zero balances. Each of these withdrawals is of size m^*. The complementary fraction gives the withdrawals that occur due to a chance meeting with the intermediary. Conditional on having money balances in $(0, \underline{m})$ then a withdrawal of size $(m^* - m)$ happens with frequency $h(m)/H(\underline{m})$.

By the same reasoning than in the $f = 0$ case, the average amount of money that an agent has at the time of withdrawal, $\underline{M}$, satisfies

$$\underline{M} = 0 \left[1 - \frac{p}{n} H(\underline{m})\right] + \left[\frac{p}{n} H(\underline{m})\right] \frac{\int_0^{\underline{m}} m\ h(m)\, dm}{H(\underline{m})} \quad .$$

As in the $f = 0$ model the relation $\underline{M} = m^* - W$ holds. Inserting the definition of $\underline{M}$ into the expression for M we obtain $\underline{M} = \frac{p}{n} M \left[1 - \frac{\int_{\underline{m}}^{m^*} m h(m) dm}{M}\right]$.

E.1 Proofs for the model with costly withdrawals

Proof of Proposition 5. Recall the 5 equation system in (11)-(15). We use repeated substitution to arrive to one non-linear equation in one unknown, namely $\underline{m}$. Equations (11) and (12) yield $V^* = R/r \;\; m^*$. Replacing V^* by this expression yields (12), so we have a system of 4 equations in 4 unknowns. We use (11) to define $A_\varphi(m^*)$ as its solution, i.e. $\varphi_m(m^*, A_\varphi(m^*)) = 0$, which yields

$$A_\varphi(m^*) = \frac{rR}{c(r + \pi)} \left[1 + \pi \frac{m^*}{c}\right]^{1 + \frac{r}{\pi}} \quad . \tag{26}$$

To solve for $A_\eta(m^*)$ we use (14) and $rV^* = Rm^*$ to get:

$$A_\eta(m^*) = \frac{r+p}{c^2}\left(Rm^* + br + p(b-f) + \frac{Rc}{r+p+\pi}\right) \tag{27}$$

Next we replace A_η and A_φ into (13) and (15) so we get two non-linear equations:

$$\begin{aligned} \eta\left(\underline{m}, (m^* R/r), A_\eta(m^*)\right) &= (m^* R/r) + f \\ \varphi\left(\underline{m}, A_\varphi(m^*)\right) &= (m^* R/r) + f \end{aligned}$$

The first equation, using (27) to substitute for $A_\eta(m^*)$, yields

$$m_1^*(\underline{m}) = \left(\frac{r+p}{R}\right)\left[\frac{c}{r+p}\left(\frac{p\,f}{c} - \frac{R}{(r+p+\pi)}\right) + \frac{\left(\frac{R}{r+p+\pi}\right)\underline{m} + b\left(1+\frac{\pi}{c}\underline{m}\right)^{-\frac{r+p}{\pi}} - f}{1-\left(1+\frac{\pi}{c}\underline{m}\right)^{-\frac{r+p}{\pi}}}\right] \tag{28}$$

Notice that for $\pi > 0$, $m_1^*(\underline{m})$ is continuous in $(0,\infty)$ and that:

$$\lim_{\underline{m}\to 0} m_1^*(\underline{m}) = +\infty \text{ and } \lim_{\underline{m}\to\infty} \frac{m_1^*(\underline{m})}{\underline{m}} = \left(\frac{r+p}{r+p+\pi}\right) < 1.$$

The second equation, using (26) to substitute for $A_\varphi(m^*)$, yields

$$m^* = \sigma(m^*, \underline{m}) \equiv \left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1+\pi\frac{m^*}{c}\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - f\frac{r}{R}. \tag{29}$$

We define $m_2^*(\underline{m})$ as the solution to $m_2^*(\underline{m}) = \sigma(m_2^*(\underline{m}), \underline{m})$. Notice that σ is increasing in m^* with

$$\frac{\partial\sigma(\underline{m},\underline{m})}{\partial m^*} = 1\ ,\ \frac{\partial\sigma(m^*,\underline{m})}{\partial m^*} > 1 \text{ for } m^* > \underline{m}, \quad \text{and } \sigma(\underline{m},\underline{m}) = \underline{m} - f\frac{r}{R}$$

so that $m_2^*(\underline{m})$ is well defined and continuous on $[0,\infty)$, that $m_2^*(0) < \infty$ and that $m_2^*(\underline{m}) > \underline{m}$ for all $\underline{m}$. Using the properties of $m_1^*(\cdot)$ and $m_2^*(\cdot)$ the intermediate value theorem implies that there is an $\underline{\hat{m}} \in (0,\infty)$ such that $m_1^*(\underline{\hat{m}}) = m_2^*(\underline{\hat{m}})$.

For $\pi < 0$ the range of the functions defined above is $[0, -\pi/c]$. By a straightforward adaptation of the arguments above one can show the existence of the solution of the two equations in this case.

Next we verify the guesses that the value function $V(m)$ is decreasing in a neighborhood of $m = 0$ and single peaked. The convexity of $V(m)$ is equivalent to showing that $A_\varphi > 0$ and $A_\eta > 0$ which can be readily established from (26) and (27) provided $b > f$. Moreover, since $A_\varphi > 0$ and $A_\eta > 0$, then $V(m)$ is strictly decreasing on $(0, m^*)$.

We extend the value function to the range (m^*, ∞). Given the values already found for V^* and A_φ we find $\bar{m}$ as the solution to $\varphi(\bar{m}, A_\varphi) = V^* + f$, i.e. $\bar{m}$ solves:

$$\left(\frac{R}{r+\pi}\right)\bar{m} + \left(\frac{c}{r}\right)^2 A_\varphi \left[1+\frac{\pi}{c}\bar{m}\right]^{-\frac{r}{\pi}} = V^* + f + \frac{Rc/(r+\pi)}{r}.$$

Now given V^* and $\bar{m}$ we find the constant $\bar{A}_\eta$ by solving $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right) = V^* + f$

$$\bar{A}_\eta = \left(\frac{r+p}{c}\right)^2\left(1+\frac{\pi}{c}\bar{m}\right)^{\frac{r+p}{\pi}}\left(V^* + f - \frac{p(V^*+f) - Rc/(r+p+\pi)}{r+p} - \frac{R}{r+p+\pi}\bar{m}\right)$$

Given V^* and $\bar{A}_\eta$ we find m^{**} as the solution of $\eta\left(m^{**}, V^*, \bar{A}_\eta\right) = V^* + b$.

Now we establish that V is strictly increasing in (m^*, m^{**}). For this notice that since $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right) = \varphi\left(\bar{m}, A_\varphi\right)$ then by inspecting the Bellman equation (7) it follows that they have the same derivative with respect to m at $\bar{m}$. Since $\varphi\left(\bar{m}, A_\varphi\right)$ is convex this derivative is strictly positive. There are two cases. If $\bar{A}_\eta$ is positive then $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right)$ is convex in this range and hence V is increasing. If $\bar{A}_\eta$ is negative then $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right)$ is concave but it is increasing since it cannot achieve a maximum since it is the sum of a linear increasing and a bounded concave function. □

Proof of Proposition 6. In Proposition 4 we show that $V(m)$ is analytical in the interval $[\underline{m},\ m^*]$. Using $V^i(\cdot)$ to denote the ith derivative of $V(\cdot)$ we can write

$$V(m) = V(m^*) + \sum_{i=1}^{\infty} \frac{1}{i!} V^i(m^*)(m - m^*)^i$$

Using $f = V(\underline{m}) - V(m^*)$ we write: $f = \sum_{i=1}^{\infty}(1/i!)V^i(m^*)(\underline{m} - m^*)^i$. Next we find an expression for $V^i(m^*)$. Differentiating the Bellman equation (4) w.r.t. m in a neighborhood of m^* yields

$$R - [r + \pi]V^1(m) = V^2(m)[c + \pi m] \tag{30}$$

evaluating at m^*, using that $V^1(m^*) = 0$ we obtain $V^2(m^*) = \frac{R}{c+\pi m^*}$. Differentiating (30) repeatedly and using induction yields

$$[r + (1+i)\ \pi]\, V^{i+1}(m) = -V^{i+2}(m)[c + \pi m] \quad \text{for } i \geq 1 \tag{31}$$

Solving the difference equation in (31) evaluated at m^* gives

$$V^{i+1}(m^*) \quad = \quad (-1)^{i-1} \frac{R}{(c + m^*\pi)^i} \Pi_{j=2}^{i}[r + j\pi] \quad \text{for } i \geq 2 \tag{32}$$

Using $V^1(m^*) = 0$, $V^2(m^*) = \frac{R}{c+\pi m^*}$ and (32) for higher order derivatives into $f = \sum_{i=1}^{\infty}(1/i!)V^i(m^*)(\underline{m} - m^*)^i$ and rearranging, yields equation (17).

For $\pi = 0$, $z = (m^* - \underline{m})/c$ solves $f/(Rc) = z^2\ \psi(z)$ where $\psi(z) = 1/2 + \sum_{k=1}^{\infty}\left(r^k z^k/(k+2)!\right)$. Since $\psi > 0$ and increasing in z then $(m^* - \underline{m})/c$ is increasing in $f/(Rc)$ with elasticity smaller than 1/2. □

Proof of Proposition 7. The proof for n is analogous to the one of the baseline model with $f = 0$. Let $\underline{t}$ be the time to deplete balances from m^* to $\underline{m}$, it solves: $(m^* - \underline{m}) = c\int_0^{\underline{t}} e^{\pi s}\ ds$, or $\underline{t} = (1/\pi)\log(1 + (m^* - \underline{m})\,\pi/c)$. The distribution of the time between withdrawals for this model has density equal to zero over the $(0, \underline{t})$ with the right truncation denoted by $\bar{t}$ which solves: $\underline{m} = c\int_0^{\bar{t}} \exp(\pi\ s)\ ds$ or$\bar{t} = (1/\pi)\log(1 + \underline{m}\pi/c)$ Thus, the expected time between withdrawals is given by: $\underline{t} + (1 - e^{-p\bar{t}})/p$. Substituting the above expressions into this formula and taking the reciprocal value yields equation (19) in the paper.

Now we turn to the derivation of $H(\underline{m})$. After each withdrawal the agent spends $\underline{t}$ units of time with $m \in (\underline{m}, m^*)$. The fundamental theorem of Renewal Theory implies that the expected time that an agent spends with $m \in (\underline{m}, m^*)$ in a period of length T converges to $n\ \underline{t}$ as $T \to \infty$. By the ergodic theorem $n\ \underline{t} = H(m^*) - H(\underline{m}) = 1 - H(\underline{m})$. Replacing the expressions for n and $\underline{t}$ yields the desired result. □

Proof of Proposition 8. By repeated differentiation of (23) (respectively (24)) it is readily verified that (21) is satisfied on the domain $(0, \underline{m})$ (respectively 22 on the domain $(\underline{m}, m^*)$). The proof is completed by verifying that the piecewise definition of H satisfies the boundary conditions that $H(0) = 0$, $H(m^*) = 1$, and that both (23) and (24) evaluated at $\underline{m}$ equal $H(\underline{m})$. □

F Testing the $f = 0$ model vs the $f > 0$ model

We examined the extent to which imposing the constraint that $f = 0$ diminishes the ability of the model to fit the data. To do so we reestimated the model letting f/c vary across province-years-households type, and compared the fit of the restricted ($f = 0$) with the unrestricted model using a likelihood ratio test.

Table V: Testing the $f > 0$ vs. $f = 0$ model

	Household w/o ATM	Household w. ATM
- Hp. $f = 0$ is rejected[c]	2%	19%

[c] Percentage of estimates where the null hypothesis of $f = 0$ is rejected by a likelihood ratio test at the 5% confidence level. Based on a comparison between the likelihood for the restricted model ($f = 0$) with the likelihood for a model where f/c is allowed to vary across province-year-type.

Table V reports the percentage of province-years-consumption cells where the null hypothesis of $f = 0$ is rejected at a 5% confidence level. It appears that only for a small fraction of cases (19% for those cells that correspond to households with ATM cards, and 2% for those without cards) there may be some improvement in the fit of the model by letting $f > 0$. We explored two approaches to estimate the $f > 0$ model. In one case we let f/c vary across province-years-household type, in the other case we fixed f/c to a common, non zero value for all province-year-types (aggregating all the cash consumption levels). We argue that while there is an improvement in the fit for a relatively small fraction of province-years by letting $f > 0$, as documented in Table V, the variables in our data set do not provide us with the type of information that would allow the parameter f to be identified. Indeed, our findings (not reported) show that when we let $f > 0$ and estimate the model for each province-year-type, the average as well as median t-statistic of the parameters $(p, b/c, f/c)$ are very low, and the average correlation between the estimates is extremely high. Additionally, there is an extremely high variability in the estimated parameters across province-years.[8] We conclude that the information in our data set does not allow us to estimate p, b/c and f/c with a reasonable degree of precision. As we explained when we introduced the model with $f > 0$, the reason to consider that model is to eliminate the extremely small withdrawals that the model with $f = 0$ implies. Hence, what would be helpful to estimate f is information on the minimum size of withdrawals, or some other feature of the withdrawal distribution.

G Pairwise Estimates (exactly identified)

This Appendix presents estimates of the structural parameters $p, b/c$ obtained using only 2 of the 4 variables (M/c, W/M, n, $\underline{M}/M$) used in the main text. As discussed in the paper, the model is then exactly identified (rather than over-identified). This approach allows us to analyze how the resulting vector of $p, b/c$ estimates varies according to the pair of variable is used.

Table VI presents a synopsis of the estimates for $p, b/c$ obtained using different pairs of observables. As for the case studied in the main body of the paper, each estimation exercise is based on about 1,500 estimation cells. The table reports the mean, the median and the standard deviation of these estimates. It appears that the estimates obtained using M/c and n are extremely similar to the ones obtained using W/M and n. Smaller values of $p, b/c$ are obtained, especially for Household with ATM, when the variables $\underline{M}/M$ and n are used. These results highlight one of the tensions that our model faces in fitting the cash patterns described in Table 1 of the paper. As can be seen from Figure 1 in the paper, fitting values of $\underline{M}/M$ of about 0.4 or lower requires values of $p^2 \cdot b/c$ that predict values of W/M that are too high compared to the data. For this reason, the estimates that do not make use of the statistics on $\underline{M}/M$ produce values for $p, b/c$

[8]The results are available upon request. In the case where f/c is fixed at the same value for all province-years, the average t-statistics are higher, but the estimated parameters still vary considerably across province-years.

that are greater. The estimates displayed in the main body of the paper (Table 3), which are based on all 4 observables, trade off the fit between the various observables. The resulting imperfect fit is used in testing for over-identification restrictions.

Table VI: $(p, b/c)$ estimates

	w ATM		wo ATM	
	Results using M/c and n			
	p	b/c	p	b/c
Mean	42	0.06	9	0.07
Median	40	0.03	7	0.05
STD	19	0.09	7	0.07
	Results using W/M and n			
	p	b/c	p	b/c
Mean	36	0.03	10	0.13
Median	34	0.02	8	0.07
STD	17	0.03	6	0.14
	Results using $\underline{M}/M$ and n			
	p	b/c	p	b/c
Mean	18	0.01	6	0.14
Median	15	0.006	5	0.06
STD	11	0.01	4	0.30

Table VII: Correlation of estimates obtained from different pairs

	Household with ATM		
	Results for p		
	$(M/c\ ,\ n)$	$(W/M\ ,\ n)$	$(\underline{M}/M\ ,\ n)$
$(M/c\ ,\ n)$	1	0.8	0.5
$(W/M\ ,\ n)$	0.8	1	0.4
	Results for b		
	$(M/c\ ,\ n)$	$(W/M\ ,\ n)$	$(\underline{M}/M\ ,\ n)$
$(M/c\ ,\ n)$	1	0.4	0.02
$(W/M\ ,\ n)$	0.4	1	0.05
	Household without ATM		
	Results for p		
	$(M/c\ ,\ n)$	$(W/M\ ,\ n)$	$(\underline{M}/M\ ,\ n)$
$(M/c\ ,\ n)$	1	0.9	0.6
$(W/M\ ,\ n)$	0.9	1	0.5
	Results for b		
	$(M/c\ ,\ n)$	$(W/M\ ,\ n)$	$(\underline{M}/M\ ,\ n)$
$(M/c\ ,\ n)$	1	0.5	0.3
$(W/M\ ,\ n)$	0.5	1	0.4

Table VII reports the correlation between the values of p (or b/c) estimated using different pairs of observables. For instance, the value of 0.8 that appears in position (1,2) of the table is the linear correlation coefficient between the estimates of b produced using $(M/c, n)$ and those obtained using $(W/M\ ,\ n)$. It appears that the correlation is positive and significantly different from zero (under the assumption of independence) for all cell except for two entries where the value if not statistically different from zero (the estimates for b for Households with ATM produced using the $\underline{M}/M$, n pair).

H Estimation with Household level heterogeneity

The estimation strategy pursued in the main body of the paper is based on two key assumptions: (i) the parameters b/c and p are the same for all households in a given cell, (ii) the variables (M/c, W/M, n, $\underline{M}/M$) are observed with a classical measurement error. An alternative estimation strategy, developed in section 5.3 of the paper, also assumes that the household variables are observed with classical measurement error, but posits that the parameters b/c and p differ for each households, and are given by a simple function of household level variables.

The next section considers yet another strategy where the estimation incorporates heterogeneity in the parameters p and b/c at the household level and NO measurement error.

H.1 Household level unobserved heterogeneity (no measurement error).

In the first case we let each household have its own pair of parameters b/c and p, assuming that we observe $(M/c, W/M, n, \underline{M}/M)$ with no error. With no need to assume a functional form for the distribution of the parameters $(p, b/c)$ this allows us to estimate the model for each household separately. Note that unless the four observables (M/c, W/M, n, $\underline{M}/M$) for a given households can be rationalized by the two parameters $(p, b/c)$the observations will be inconsistent for this household. To address this stochastic singularity (that comes from having four observables, no measurement error and only two parameters), we estimate the model using only two observables: $(M/c, n)$, the two variables for which we have more observations. To be concrete we apply this estimation strategy to a subset of households: those in a province (e.g. province # 1 in our sample: Turin), a year (say 1993 or 2002), those with or without ATM cards, and those from a third-tile of the cash consumption distribution. We label the estimates for such set of households as *HH estimates* and compare them with two *Cell estimates* obtained using the mean of the observables at the cell level. One cell-estimate uses the mean of two observables $(M/c, n)$, so that the only difference with the *HH estimates* is that it produces one parameter vector for the whole cell instead of one parameter vector for each households. The other cell-estimate uses the mean of the cell for four observables (M/c, W/M, n, $\underline{M}/M$) and ASSUMES MEASUREMENT ERROR, which is the benchmark case considered in the paper. This allows to compare the effect of using two vs four observables.

The HH estimates produce a distribution of the parameters $(p, b/c)$ for each cell. In the tables below, each of the cell at which we estimate the model corresponds to a different column. The estimation at the household level are in the top panel, where we report its mean, median and standard deviation of the estimated parameters across households. Similar results were obtained for other provinces and HH types (not reported for reasons of space).

We detect a very large heterogeneity on the values of the parameters, reflecting the large variability of M/c and n across households in a given cell. Indeed the mean value of the parameter b/c is greatly affected by some of the extreme values estimated. The bottom panel of the tables displays the estimated value of the parameters using the means at the cell level, labeled as "2 var" and "4 vars". The median value of the estimated parameters at the household level is close to the value estimated using the cell means, especially so when it is compared with the estimation using the same two observables.

In this section we have compared two extreme assumptions. One assumption, used for our benchmark estimates, is that all heterogeneity within a cell is accounted for measurement error. The large dispersion of the observables $(M/c, n)$ in a given cell is then accounted for by a the large value of the variance of the measurement error σ_j^2. The other assumption is that all the heterogeneity within a cell is due to differences in the parameters across agents. In this case the large dispersion in the observables $(M/c, n)$ in a given cell is accounted for by the large heterogeneity of the estimated parameters $(p, b/c)$. In summary, the median of

Table VIII: HH with ATM; $(p, b/c)$ estimates, Turin, 1993

Cash exp. group	low	med.	high	low	med.	high
	Parameter p			Parameter $b/c_{day} \cdot 100$		
	HH estimates					
Mean	29	26	46	10^{11}	13	10^3
Median	27	17	33	7	4	3
stds.	24	35	53	10^{13}	31	10^5
	Cell mean estimates					
– 2 vars	28	17	37	7	3	2
– 4 vars (as in paper)	13	20	31	5	2	2

Table IX: HH with ATM; $(p, b/c)$ estimates, Turin, 2002

Cash exp. group	low	med.	high	low	med.	high
	Parameter p			Parameter $b/c_{day} \cdot 100$		
	HH estimates					
Mean	38	23	37	10^{12}	40	25
Median	22	16	33	47	8	5
stds.	51	22	28	10^{13}	215	115
	Cell mean estimates					
– 2 vars	27	20	36	26	7	5
– 4 vars (as in paper)	11	12	19	8	3	2

the estimates that allow for household level heterogeneity is remarkably close to the value that is estimated using the mean of the cell level. However, the dispersion on the estimated parameters $p, b/c$ is huge, so much that the mean estimated values for the parameter b/c are meaningless. We find that this heterogeneity is too large to reflect purely differences across households. Hence we do not pursue further the estimation strategy at the household level. An alternative estimation strategy, pursued in the section 5.3 of the paper, combines both measurement error and heterogeneity.

I Estimation under alternative cell definitions

This appendix reports the estimation results of the model with random free withdrawals obtained under five alternative aggregation and selection of the raw data.

The baseline aggregation used in the estimates of Section 5 includes all households with a deposit account for whom the survey data are available (see the paper for details). The elementary household data were aggregated at the province-year-household type (ATM/noATM and 3 consumption groups), providing us with a total of about 1,800 observations per type of withdrawal technology (ATM, no ATM) to be fitted (103 provinces * 6 years * 3 consumption groups), each one based on approximately 13 elementary household observations. Four additional aggregations of the data were explored. Table X provides a quick synopsis that is helpful to compare the results obtained from our benchmark specification (reported in column 4 for ease of comparison) with the ones produced by those alternatives.

The first alternative aggregation of the data, reported in column 1 of Table X, differs from the baseline case in that it does not split households according to their consumption level. This increase by about 3 times the number of elementary household observations used for the estimate of $(p, b/c)$ in a given province-year-household type. The value of the point estimates is close the one obtained in the baseline exercise, though the greater number of underlying observation increases the statistical significance of the estimates.

Table X: Estimation outcomes over five different datasets

Dataset[a]:	py (raw)	py (filt.)	ryc (filt.)	pyc (raw)	pyc (filt)
			Households with ATM		
N. of estimates	576	563	532	1,654	1,454
Mean N. of HH per est.	39	17	16	14	6
% of estimates where:					
- Hp. $f = 0$ rejected	40	33	42	19	17
- $F(\theta, x) < 4.6$	42	47	38	57	60
Mean estimate of p	22	29	27	22	29
mean t-stat	4.9	4.4	4.4	3.1	3.0
Corr. w. Bank Branches[b]	0.1	0.0	0.0	0.1	0.1
Mean estimate of $b/c \cdot 100$	2.6	2.5	2.5	4.0	3.8
mean t-stat	4.5	3.3	3.5	2.8	2.3
Corr. w. Bank Branches[b]	-0.2	-0.2	-0.3	-0.2	-0.2
			Households without ATM		
N. of estimates	550	538	535	1,539	1,411
Mean N. of HH per est.	30	14	13	11	5
% of estimates where:					
- Hp. $f = 0$ rejected	9	6	3	2	1
- $F(\theta, x) < 4.6$	49	66	70	64	74
Mean estimate of p	7	7	7	8	8
mean t-stat	3.7	3.1	3.0	2.4	2.1
Corr. w. Bank Branches[b]	0.0	0.0	-0.1	0.0	0.1
Mean estimate of $b/c \cdot 100$	6.7	6.2	5.8	7.7	7.4
mean t-stat	4.2	3.3	3.3	2.7	2.3
Corr. w. Bank Branches [b]	-0.3	-0.2	-0.3	-0.3	-0.2

Notes: Sample statistics computed on the distribution of the estimates after trimming the $(p, b/c)$ distribution tails of the highest and lowest percentiles (1 per cent from each tail). The variable b/c is measured as a percentage of the daily cash expenditure.

-[a] The labels XYZ on this line denote the type of aggregation applied to the elementary household data: X refers to whether data were aggregated at the province (p) or region (r) level; Y indicates that data were aggregated at the year level, Z (either empty or equal to c) indicates whether households were clustered within the relevant observation unit, e.g. in each province-year (py), on the basis of their cash expenditure level (3 bins were considered for the province-year dataset, 5 bins for the region-year dataset). The label (raw/filt.) indicates whether the aggregation is based on the raw data or on a filtered dataset which excludes households who receive more than 50% of income in cash and/or violate the cash-holdings identity by more than 200%.

-[b] Correlation coefficient between the estimated values of $(p, b/c)$ and the number of bank branches per capita measured at the province level. All variables are measured in logs.

Two alternative aggregations of the data exclude households who receive more than 50% of their income in cash or violate the cash flow identity of equation (33) by more than 200%. This choice removes households for whom cash inflows are an important source of replenishment (as this channel is ignored by our baseline model) and observations affected by large measurement error. This selection criterion roughly halves the number of elementary observations. The estimation results obtained from these data when one or three consumption groups are considered (columns 2 and 5 of Table X, respectively) are extremely similar to the ones of the baseline case (column 4).

The last experiment that we report involves aggregation of the household data at the regional, rather than province, level (a region is a geographical unit which contains several provinces (there are 103 provinces and 20 regions in Italy). This allows us to consider a finer grid of consumption classes, namely 5 for the instance reported in the third column of the Table, thus increasing the mean number of elementary observations used in each estimation cell. Again, as the table shows, the results are similar to the ones produced by the other approaches.

J Cash-Flow Identity: theory and evidence

We derive the following relationship

$$c = nW - \pi M \tag{33}$$

between the average (real) cash balances M, average (real) withdrawal amount, W, average (real) consumption flow c, average number of withdrawals n per unit of time, and the inflation rate π for a (large) class of cash management policies. In what follows we fixed a particular path and denote the real cash balances at time t by $m(t)$, and let τ_i be the times at which there are withdrawals for this sample path and w_i the corresponding withdrawals amounts. In between withdrawals cash balances satisfy

$$\frac{dm(t)}{dt} = -c - m(t)\,\pi$$

At times $t = \tau_i$, a withdrawal of size w_i occurs, defined as an upward jump on m :

$$w_i \equiv \lim_{t \downarrow \tau_i} m(t) - \lim_{t \uparrow \tau_i} m(t) > 0.$$

Thus we have that

$$m(t) = m(0) - \int_0^T (c + \pi m(s))\, ds + \sum_{i=1}^{N(T)} w_i$$

where $N(T)$ denotes the number of (upward) jumps up to time T in the path:

$$N(T) \equiv \{N : \ \tau_N \leq T \leq \tau_{N+1}\}.$$

Dividing by T and rearranging:

$$\frac{m(t) - m(0)}{T} = -c - \pi \frac{1}{T}\int_0^T m(s)\, ds + \left[\frac{N(T)}{T}\right]\left[\frac{1}{N(T)}\sum_{i=1}^{N(T)} w_i\right]$$

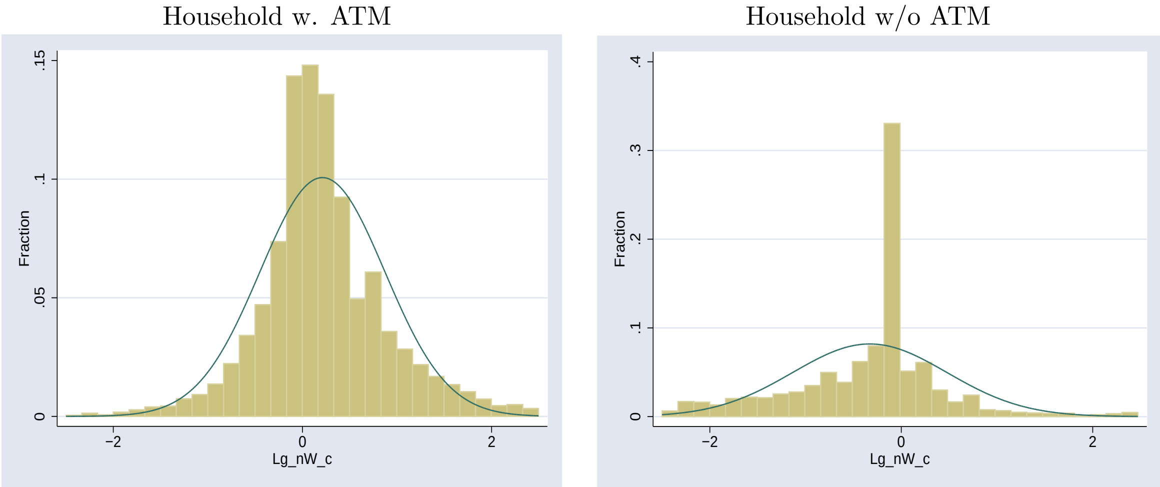
Defining :

$$M \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T m(s)\, ds, \quad n \equiv \lim_{T\to\infty} \frac{N(T)}{T}, \text{ and } W \equiv \lim_{T\to\infty} \frac{1}{N(T)}\sum_{i=1}^{N(T)} w_i$$

where $M,\ n,\ W$ are the average money balances, average number of withdrawals per unit of time, and the average amount of withdrawal. Assuming that, for almost all paths, the limits $M,\ n$ and W are well defined, and that the process is ergodic, so that these time averages converge to the unconditional expectations for almost all paths, we obtain equation 33. In all the models we analyze, these limits exist and coincide for all paths as a consequence of basic results on renewal theory, but of course their validity is much more general.

An illustration of the extent of the measurement error can be derived by assuming that the data satisfy the identity in (33). Figure III reports a histogram of the logarithm of $n\ (W/c) - \pi\,(M/c)$ for each type of household. In the absence of measurement error, all the mass should be located at zero. It is clear that the

Figure III: Measurement error: deviation from the cash flow identity



data deviate from this value for many households.[9] At least for households with an ATM card, we view the histogram as well approximated by a normal distribution (in log scale).

K Maximum likelihood and the criterion function

Let

$$\begin{aligned} L(\theta;x) &= \prod_{j=1}^{J}\prod_{i=1}^{n_j}\frac{1}{\left(2\pi\sigma_j^2\right)^{1/2}}\exp\left(-\frac{1}{2}\left[\frac{x_i^j - f^j(\theta)}{\sigma_j}\right]^2\right) \\ &= \prod_{j=1}^{J}\left(2\pi\sigma_j^2\right)^{-n_j/2}\times\prod_{j=1}^{J}\prod_{i=1}^{n_j}\exp\left(-\frac{1}{2}\left[\frac{x_i^j - f^j(\theta)}{\sigma_j}\right]^2\right) \end{aligned}$$

or

$$\ln L(\theta;x) = -\frac{1}{2}\sum_{j=1}^{J} n_j \ \log(2\pi) - \frac{1}{2}\sum_{j=1}^{J} n_j \ \log\left(\sigma_j^2\right) - \frac{1}{2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\left[\frac{x_i^j - f^j(\theta)}{\sigma_j}\right]^2$$

where $L(\theta;x)$ is the likelihood function, x_i^j the i observation of the j variable, and $f^j(\theta)$ the prediction of the model of the j variable for the parameter vector θ. The number n_j is the size of the sample of the variable j. The idea behind this is that the variable x_i^j is measured with error ε_i^j which is normal with mean zero and variance σ_j^2 so that

$$x_i^j = f^j(\theta) + \varepsilon_i^j.$$

[9]Besides measurement error in reporting, which is important in this type of survey, there is also the issue of whether households have an alternative source of cash. An example of such as source occurs if households are paid in cash. This will imply that they do require fewer withdrawals to finance the same flow of consumption or, alternatively, that they effectively have more trips per periods.

It is assumed that the errors are independent across variables as well as observations. Define

$$F(\theta; x) \equiv \sum_{j=1}^{J} \left(\frac{n_j}{\sigma_j^2}\right) \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2$$

as the criterion function that we minimize. Let σ_j be the sample analog, i.e.

$$\sigma_j^2 \equiv var\left(x^j\right) = \frac{1}{n_j} \sum_{i=1}^{n_j} \left[x_i^j - \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j\right)\right]^2$$

Proposition 9. *The likelihood is related to the criterion function as follows:*

$$\ln L(\theta; x) \quad = \quad -\frac{1}{2} \sum_{j=1}^{J} n_j \ \log(2\pi) - \frac{1}{2} \sum_{j=1}^{J} n_j \ \log\left(\sigma_j^2\right) - \frac{1}{2} \sum_{j=1}^{J} n_j - \frac{1}{2} F(x; \theta)$$

Proof. Write

$$\begin{aligned}
& \frac{1}{n_j} \sum_{i=1}^{n_j} \left[x_i^j - f^j(\theta)\right]^2 = \frac{1}{n_j} \sum_{i=1}^{n_j} \left(x_i^j\right)^2 + \frac{1}{n_j} \sum_{i=1}^{n_j} \left(f^j(\theta)\right)^2 - 2\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j f^j(\theta) \\
= \quad & \frac{1}{n_j} \sum_{i=1}^{n_j} \left(x_i^j\right)^2 + \frac{1}{n_j} \sum_{i=1}^{n_j} \left(f^j(\theta)\right)^2 - 2\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j f^j(\theta) \\
= \quad & \frac{1}{n_j} \sum_{i=1}^{n_j} \left(x_i^j\right)^2 + \left(f^j(\theta)\right)^2 - 2f^j(\theta) \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j\right) \\
= \quad & \frac{1}{n_j} \sum_{i=1}^{n_j} \left(x_i^j\right)^2 - \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j\right)^2 + \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j\right)^2 + \left(f^j(\theta)\right)^2 - 2f^j(\theta) \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j\right) \\
= \quad & \frac{1}{n_j} \sum_{i=1}^{n_j} \left(x_i^j\right)^2 - \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j\right)^2 + \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2 \\
= \quad & var\left(x^j\right) + \left(\frac{1}{n_j} \sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2
\end{aligned}$$

Thus

$$\begin{aligned}
& \ln L(\theta;x) + \frac{1}{2}\sum_{j=1}^{J} n_j \ \log(2\pi) + \frac{1}{2}\sum_{j=1}^{J} n_j \ \log\left(\sigma_j^2\right) \\
= & -\frac{1}{2}\sum_{j=1}^{J} \frac{n_j}{(\sigma_j)^2}\left\{\frac{1}{n_j}\sum_{i=1}^{n_j}\left[x_i^j - f^j(\theta)\right]^2\right\} \\
= & -\frac{1}{2}\sum_{j=1}^{J} \frac{n_j}{(\sigma_j)^2}\left\{var\left(x^j\right) + \left(\frac{1}{n_j}\sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2\right\} \\
= & -\frac{1}{2}\sum_{j=1}^{J}\left\{n_j\frac{var\left(x^j\right)}{\sigma_j^2} + \left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{1}{n_j}\sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2\right\} \\
= & -\frac{1}{2}\sum_{j=1}^{J}\left\{n_j + \left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{1}{n_j}\sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2\right\} \\
= & -\frac{1}{2}\sum_{j=1}^{J} n_j - \frac{1}{2}\sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{1}{n_j}\sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)^2 \\
= & -\frac{1}{2}\sum_{j=1}^{J} n_j - \frac{1}{2}F(x;\theta)
\end{aligned}$$

Corollary I: The score of the likelihood. Let M be the dimensionality of θ. The n element of the score is given by

$$\begin{aligned}
s_n(\theta;x) \ \equiv & \ \frac{\partial \log L(\theta;x)}{\partial\theta_n} = -\frac{1}{2}\frac{\partial F(x;\theta)}{\partial\theta_n} \\
= & -\frac{1}{2}\frac{\partial F(x;\theta)}{\partial\theta_n} = -\sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{1}{n_j}\sum_{i=1}^{n_j} x_i^j - f^j(\theta)\right)\frac{\partial f^j(\theta)}{\partial\theta_n}
\end{aligned}$$

K.1 Computing the Information matrix

Corollary II: Information matrix. Let M be the dimensionality of θ. The n,m of the $M\times M$ information matrix $I(\theta)$ is defined as:

$$I_{n,m}(\theta) = E\left[\frac{\partial \log L(\theta;x)}{\partial\theta_n}\frac{\partial \log L(\theta;x)}{\partial\theta_m}\right] = E\left[s_n(\theta,x)\, s_m(\theta,x)\right]$$

which in our case becomes

$$\begin{aligned}
I_{n,m}(\theta) &= E\left[s_n(\theta,x)\, s_m(\theta,x)\right] \\
&= E\left\{\left[\sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{1}{n_j}\sum_{i=1}^{n_j}x_i^j - f^j(\theta)\right)\frac{\partial f^j(\theta)}{\partial\theta_n}\right]\left[\sum_{j'=1}^{J}\left(\frac{n_j}{\sigma_{j'}^2}\right)\left(\frac{1}{n_{j'}}\sum_{i=1}^{n_{j'}}x_i^{j'} - f^{j'}(\theta)\right)\frac{\partial f^{j'}(\theta)}{\partial\theta_m}\right]\right\} \\
&= \sum_{j'=1}^{J}\sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{n_{j'}}{\sigma_{j'}^2}\right)E\left\{\left(\frac{1}{n_j}\sum_{i=1}^{n_j}x_i^j - f^j(\theta)\right)\left(\frac{1}{n_{j'}}\sum_{i=1}^{n_{j'}}x_i^{j'} - f^{j'}(\theta)\right)\right\}\frac{\partial f^{j'}(\theta)}{\partial\theta_m}\frac{\partial f^j(\theta)}{\partial\theta_n} \\
&= \sum_{j'=1}^{J}\sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{n_{j'}}{\sigma_{j'}^2}\right)E\left\{\left(\frac{1}{n_j}\sum_{i=1}^{n_j}\varepsilon_i^j\right)\left(\frac{1}{n_{j'}}\sum_{i=1}^{n_{j'}}\varepsilon_i^{j'}\right)\right\}\frac{\partial f^{j'}(\theta)}{\partial\theta_m}\frac{\partial f^j(\theta)}{\partial\theta_n} \\
&= \sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left(\frac{n_j}{\sigma_j^2}\right)\left\{\left[\frac{1}{n_j}\right]\sigma_j^2\right\}\frac{\partial f^{j'}(\theta)}{\partial\theta_m}\frac{\partial f^j(\theta)}{\partial\theta_n} \\
&= \sum_{j=1}^{J}\left(\frac{n_j}{\sigma_j^2}\right)\left[\frac{\partial f^j(\theta)}{\partial\theta_m}\frac{\partial f^j(\theta)}{\partial\theta_n}\right]
\end{aligned}$$

Corollary III: Akaike. For the Akaike criteria, let's $F_k(x;\theta)$ be the objective function with k variables and thus

$$\begin{aligned}
\ln L_k(\theta;x) &= -\frac{1}{2}\sum_{j=1}^{J}n_j\ \log(2\pi) - \frac{1}{2}\sum_{j=1}^{J}n_j\ \log\left(\sigma_j^2\right) - \frac{1}{2}\sum_{j=1}^{J}n_j \\
&\quad -\frac{1}{2}F_k(x;\theta).
\end{aligned}$$

Then we compare:

$$A(k,\theta) = -2\ln L_k(\theta;x) + 2k + \frac{2k(k+1)}{(n-k-1)}$$

where $n = \sum_{j=1}^{J} n_j$. Since the first three terms of the $\ln L$ do not depend on k, we can compare:

$$\tilde{A}(k,\theta) = F_k(x;\theta) + 2k + \frac{2k(k+1)}{(n-k-1)}.$$

In our case, we have that k is either 6, (p,b,σ_j) for $j=1,...,4$ or k is 7 for (p,b,f,σ_j) for $j=1,...,4$. The typical values of n are about 80, since we have $n \cong 4\ \times 20$, i.e. about 20 observations for each of the four variables.

L Technical Notes

L.1 Alternative data sources for ATM withdrawals

In this appendix we compare data on average ATM withdrawals drawn from two sources: our households survey data (*SHIW*) and the data drawn from banks' records as reported in the ECB Blue Book (2006). Table 12.1a in the bluebook reports the total number of cash withdrawals at ATMs in a year. Table 13.1a gives the total value of cash withdrawals at ATMs in a year. The average withdrawal computed as the ratio of these two numbers for the years 2001, 2002 and 2004 is 162, 205 and 169 euros, respectively (these years are the closest to those of the *SHIW* survey years). In the household survey we compute the analogue statistics for the years 2000, 2002 and 2004 obtaining 177, 185 and 205 euros, respectively. For each year the latter statistics were computed as the ratio between the sum across households of the amount of cash withdrawn from ATMs and the sum across households of the number of withdrawals from ATMs. For each household, the total amount of cash withdrawn from ATM was given by the average ATM withdrawal times the number of ATM withdrawals. These statistics differ from the statistics on W reported in Table 1 in the paper for three reasons. First because even for households with ATM card W includes withdrawals done at the bank desk (which are larger on average). Second W is measured in 2004 euros. Third W reports the average withdrawal per household, so the weighting is different.

L.2 Solution for the Value Functions ODEs

ODEs of the form:

$$f(x) = a_0 + a_1 x + (a_2 + a_3 x) f'(x)$$

appear in this paper as Bellman equations. Their solution is

$$f(x) = A_0 + A_1 x + A \left[1 + \frac{A_2}{A_3} x\right]^{A_3}$$

To see this notice that

$$f'(x) = A_1 + A\ A_3 \left(\frac{A_2}{A_3}\right) \left[1 + \frac{A_2}{A_3} x\right]^{(A_3 - 1)}$$

which requires:

$$\begin{aligned} & A_0 + A_1 x + A \left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{A_3} \\ = \;& a_0 + a_1 x + (a_2 + a_3 x) \left(A_1 + A\ A_3 \left(\frac{A_2}{A_3}\right) \left[1 + \frac{A_2}{A_3} x\right]^{(A_3 - 1)} \right) \end{aligned}$$

Solving the system of equations defined by the previous equality yields:

$$A_0 = a_0 + a_2 a_1 / (1 - a_3) \quad A_1 = a_1 / (1 - a_3) \quad A_2 = 1/a_2 \quad A_3 = 1/a_3$$

□

L.3 Expressions for the model with $f = 0$ when $\pi = 0$

This appendix collects the expression that are obtained in the case of $\pi = f = 0$. In most cases they are obtained using L'Hopital rule in the corresponding formulas for the general case. The expression for m^* is

$$\exp\left(\frac{m^*}{c}(r+p)\right) = 1 + \frac{m^*}{c}(r+p) + (r+p)^2 \frac{b}{cR} \;. \tag{34}$$

and the expression for the value function is

$$V(m) = \left[\frac{pV^*(r+p) - Rc}{(r+p)^2}\right] + \left[\frac{R}{r+p}\right] m + \left(\frac{c}{r+p}\right)^2 A \exp\left(-\frac{r+p}{c} m\right) \;.$$

The expression for the expected number of trips per unit of time n is

$$n(m^*; c, 0, p) = \frac{p}{1 - e^{-m^* \frac{p}{c}}} \tag{35}$$

The expression for the density of the distribution of real cash balances is

$$h(m) = \frac{\frac{p}{c}\exp\left(\frac{mp}{c}\right)}{\exp\left(\frac{m^* p}{c}\right) - 1} \tag{36}$$

The expression for aggregate money balances

$$M = c\left[\frac{1}{1-e^{-\frac{p}{c}m^*}}\frac{m^*}{c} - 1/p\right] \;. \tag{37}$$

L.4 Expressions for the model with $f > 0$ when $\pi = 0$

This appendix collects the expression that are obtained in the case of $f > 0$ and $\pi = 0$. In most cases they are obtained using L'Hopital rule in the corresponding formulas for the general case. For a given V^* and $0 < \underline{m} < \bar{m}$ the solution of $V(m)$ for $m \in (\underline{m}, \bar{m})$ is given by:

$$\begin{aligned} V(m) &= \varphi(m, A_\varphi) \equiv \\ &\equiv \left[\frac{-Rc}{r^2}\right] + \left[\frac{R}{r}\right] m + \left(\frac{c}{r}\right)^2 A_\varphi \exp\left(-\frac{r}{c}m\right) \;. \end{aligned}$$

and

$$\begin{aligned} V(m) &= \eta(m, V^*, A_\eta) \equiv \\ &\left[\frac{p(V^*+f)(r+p) - Rc}{(r+p)^2}\right] + \left[\frac{R}{r+p}\right] m + \left(\frac{c}{r+p}\right)^2 A_\eta \exp\left(-\frac{r+p}{c}m\right) \;. \end{aligned}$$

for $m \in (0,\ \underline{m})$ or $m \in (\bar{m},\ m^{**})$. The range of inaction $(m^* - \underline{m})$ is given by:

$$\frac{f\ c}{R} = [m^* - \underline{m}]^2 \left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[(m^* - \underline{m})\ \frac{r}{c}\right]^{j-2}\right) \tag{38}$$

Calculations for $m^* - \underline{m}$ for the case of $\pi = 0$. To see how we obtain the result for $\pi = 0$, start with the expression for $z^* = m^* - \underline{m}$:

$$z^* = \frac{1}{r/c}\left(\exp\left[z^*\ \frac{r}{c}\right] - 1\right) - f\frac{r}{R}.$$

Write this expression as:

$$\exp\left[z^*\ \frac{r}{c}\right] \quad = \quad 1 + \left[z^*\ \frac{r}{c}\right] + \left[z^*\ \frac{r}{c}\right]^2 \left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[z^*\ \frac{r}{c}\right]^{j-2}\right)$$

hence

$$\frac{f\ c}{R} = [m^* - \underline{m}]^2 \left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[z^*\ \frac{r}{c}\right]^{j-2}\right)$$

The CDF for $\pi = 0$.

For $m \in (0, \underline{m})$ we have

$$\begin{aligned}
H(m) &= \frac{A_0}{p/c} \exp(pm/c) - B_0 && (39) \\
H(\underline{m}) &= \frac{1 - \exp(-p(\underline{m}/c))}{p(m^* - \underline{m})/c + 1 - \exp(-p(\underline{m}/c))} \\
A_0 &= \frac{H(\underline{m})(p/c)}{[\exp(p\underline{m}/c) - 1]} && (40) \\
B_0 &= \frac{A_0}{p/c} && (41)
\end{aligned}$$

For $m \in (\underline{m}, m^*)$ we have

$$\begin{aligned}
H(m) &= \frac{A_1}{\pi} \log\left(1 + \pi \frac{m}{c}\right) - B_1 && (42) \\
[1 - H(\underline{m})] &= \frac{p(m^* - \underline{m})/c}{p(m^* - \underline{m})/c + 1 - \exp(-p\underline{m}/c)} \\
A_1 &= \frac{1 - H(\underline{m})}{(m^* - \underline{m})/c} && (43) \\
B_1 &= A_1 m^*/c - 1 && (44)
\end{aligned}$$

The average money holdings and withdrawals for $\pi = 0$

$$\begin{aligned}
M &= m^* - \frac{A_0}{(p/c)} \left\{ \frac{[\exp(p\underline{m}/c) - 1]}{(p/c)} - \underline{m} \right\} && (45) \\
&\quad - \frac{A_1}{c} \left((m^*)^2 - (\underline{m})^2 \right) + [A_1 m^*/c - 1](m^* - \underline{m})
\end{aligned}$$

where A_0, A_1 and B_1 are given in (40),(43) and (44).

If $\pi = 0$ the average withdrawal W is given by:

$$W = m^* \left[1 - \frac{p}{n} H(\underline{m})\right] + \left[\frac{p}{n} H(\underline{m})\right] \frac{\int_0^{\underline{m}} (m^* - m) h(m) \, dm}{H(\underline{m})} \qquad (46)$$

where

$$\frac{\int_0^{\underline{m}} (m^* - m) h(m) \, dm}{H(\underline{m})} = m^* - \underline{m} - \frac{\frac{[\exp(p\underline{m}/c) - 1]}{(p/c)} - \underline{m}}{\exp(p\underline{m}/c) - 1}$$

L.5 Expression (with derivation) for M when $f > 0$

$$\int_0^{\underline{m}} m h(m) \, dm = \left[H(\underline{m}) \underline{m} - H(0) 0 - \int_0^{\underline{m}} H(m) \, dm \right]$$

where

$$\int_0^{\underline{m}} H(m) \, dm = \int_0^{\underline{m}} \frac{A_0}{p/c} \left(1 + \pi \frac{m}{c}\right)^{\frac{p}{\pi}} dm - B_0(\underline{m}) = \frac{A_0}{p/c} \left[\frac{\left(1 + \frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1} - 1}{(p + \pi)/c} - \underline{m} \right]$$

and

$$\int_{\underline{m}}^{m^*} mh(m)\,dm = \left[m^* - H(\underline{m})\,\underline{m} - \int_{\underline{m}}^{m^*} H(m)\,dm\right]$$

$$\int_{\underline{m}}^{m^*} H(m)\,dm = \int_{\underline{m}}^{m^*} \frac{c}{\pi} A_1 \log\left(1+\pi\frac{m}{c}\right) dm - B_1\left(m^* - \underline{m}\right)$$

where

$$\int_{\underline{m}}^{m^*} \log\left(1+\pi\frac{m}{c}\right) dm = \frac{c}{\pi}\left(1+\pi\frac{m}{c}\right)\left[\log\left(1+\frac{\pi}{c}m\right)-1\right]|_{\underline{m}}^{m^*}$$

Hence

$$\begin{aligned}\int_{\underline{m}}^{m^*} H(m)\,dm &= A_1\left(\frac{c}{\pi}\right)^2\left\{\left(1+\frac{\pi}{c}m^*\right)\left[\log\left(1+\pi\frac{m^*}{c}\right)-1\right]\right.\\ &\left.-\left(1+\frac{\pi}{c}\underline{m}\right)\left[\log\left(1+\frac{\pi}{c}\underline{m}\right)-1\right]\right\} - B_1\left(m^*-\underline{m}\right)\end{aligned}$$

Thus

$$\begin{aligned} M &= m^* - \int_0^{\underline{m}} H(m)\,dm - \int_{\underline{m}}^{m^*} H(m)\,dm \\ &= m^* - \frac{c}{p}A_0\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}\right] \\ &-A_1\left(\frac{c}{\pi}\right)^2\left\{\left(1+\frac{\pi}{c}m^*\right)\left[\log\left(1+\pi\frac{m^*}{c}\right)-1\right] - \left(1+\frac{\pi}{c}\underline{m}\right)\left[\log\left(1+\frac{\pi}{c}\underline{m}\right)-1\right]\right\} \\ &+\left(m^*-\underline{m}\right)\left(\frac{c}{\pi}A_1\log\left(1+\pi\frac{m^*}{c}\right)-1\right)\end{aligned}$$

where

$$A_0 = \frac{p}{c}\frac{1}{\left[\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{p}{\pi}}-1\right]}H(\underline{m})$$

$$A_1 = \frac{\left(1-H(\underline{m})\right)(\pi/c)}{\log\left(1+\pi\frac{m^*}{c}\right)-\log\left(1+\frac{\pi}{c}\underline{m}\right)}$$

L.6 Expression (with derivation) for W when $f > 0$

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The expression

$$\frac{\int_0^{\underline{m}}\left(m^*-m\right)h(m)\,dm}{H(\underline{m})}$$

is the expected withdrawal conditional on being done by an agent with $m > 0$, or conditional on being a withdrawal that happens due to a chance meeting with the intermediary.

$$\int_0^{\underline{m}}\left(m^*-m\right)h(m)\,dm = m^*H(\underline{m}) - \int_0^{\underline{m}} mh(m)\,dm$$

$$\int_0^{\underline{m}} m h(m)\, dm = \underline{m} H(\underline{m}) - \int_0^{\underline{m}} H(m)\, dm$$

with

$$\int_0^{\underline{m}} H(m)\, dm = \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}\right]$$

Thus

$$\int_0^{\underline{m}} (m^* - m) h(m)\, dm = (m^* - \underline{m}) H(\underline{m}) + \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}\right]$$

$$\left(\frac{A_0}{p/c}\right)/H(\underline{m}) = \frac{1}{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}}-1}$$

so

$$\begin{aligned}
\frac{\int_0^{\underline{m}} (m^* - m) h(m)\, dm}{H(\underline{m})} &= (m^* - \underline{m}) + \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}\right] \\
&= (m^* - \underline{m}) + \frac{\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}}{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}}-1}
\end{aligned}$$

L.7 Solving for b and f

Here we describe how to find b and f given $(m^*,\ \underline{m},\ r, \pi, R)$ For convenience we rewrite equation (29) for $m_2^*(\cdot)$:

$$m^* = \left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1+\pi\frac{m^*}{c}\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - f\frac{r}{R}$$

to find f. It is given by:

$$f = \frac{\left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1+\pi\frac{m^*}{c}\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - m^*}{r/R}$$

Given f and $(m^*,\ \underline{m},\ r, \pi, R, p)$ use equation (28) for $m_1^*(\cdot)$:

$$m^* = \frac{\left(\frac{c}{r+p}\right)\left[\frac{p\ f}{c} - \frac{R}{(r+p+\pi)}\right]}{\left(\frac{R}{r+p}\right)} + \frac{\left[\frac{R}{r+p+\pi}\right]\underline{m} + b\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} - f}{\left(\frac{R}{r+p}\right)\left[1-\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right]}$$

to find b. It is given by

$$b = \frac{\left(m^* - \frac{\left(\frac{c}{r+p}\right)\left[\frac{p\ f}{c} - \frac{R}{(r+p+\pi)}\right]}{\left(\frac{R}{r+p}\right)}\right)\left(\frac{R}{r+p}\right)\left[1-\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right] - \left[\frac{R}{r+p+\pi}\right]\underline{m} + f}{\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}}$$

L.8 Weights used in the estimation

Table XI displays the average weights N_j/σ_j^2 used in estimation, the average N_j (across provinces and years), and the estimated value of σ_j^2. The latter are estimated as the variance of the residual of a regression of each of the j variables at the household level against dummies for each province-year combination (separate regressions are used for households with and without ATM cards).

Table XI: Weights used in estimation

	$log(M/c)$	$log(W/M)$	$log(n)$	$log(\underline{M}/M)$
	Households with ATM			
Average weight (N_j/σ_j^2)	30	17	22	14
Variance (σ_j^2)	0.46	0.42	0.53	0.82
Average # of Households in province-year-consumption cell (N_j)	13.5	6.3	12	9.5
	Households without ATM			
Average weight (N_j/σ_j^2)	26	14	12	11
Variance (σ_j^2)	0.41	0.51	0.62	0.82
Mean # of Households in province-year-consumption cell (N_j)	10.7	7.4	7.6	7.6

Notes: There is a total of 3,189 estimation cells (the available observations of the cartesian product of 6 years, 103 provinces, ATM ownership and 3 consumption groups).

L.9 Identification of p, b using $(W/M\ , n)$ or $(\underline{M},\ n)$

Figures IV and V supplement the information of Figure 2 in the paper, showing how one can identify the values of $p, b/c$ using either the pair $(W/M, n)$ or the pair $(\underline{M}/M, n)$ as an alternative to $(M/c, n)$. As for figure 2 in the main text, each dot represents the mean across households of a given province-year-type, its size proportional to the size of the province. Blue dots denote households with ATM card. The thick black lines represent the theoretical loci predicted by the model discussed in Section 5.1 of the paper.

L.10 Decomposition of the cost of financing c

Let $v(R, \pi, p, b/c)/c$ be the per unit cost of financing cash purchases given the vector $(R, \pi, p, b/c)$, which is then expressed in number of days of cash purchases. To measure the *Reduction in cost in # cash days* in Table XII we define

$$\Delta v_{t,i} \equiv v(R_{0,i}, \pi_{0,i}, p_{0,i}, (b/c)_{0,i})/c_{0,i} - v(R_{t,i}, \pi_{t,i}, p_{t,i}, (b/c)_{t,i})/c_{t,i} \tag{47}$$

$$\Delta v_{t,i,(p,b)} \equiv v(R_{0,i}, \pi_{0,i}, p_{0,i}, (b/c)_{0,i})/c_{0,i} - v(R_{0,i}, \pi_{0,i}, p_{t,i}, (b/c)_{t,i})/c_{t,i} \tag{48}$$

$$\Delta v_{t,i,(R,\pi)} \equiv v(R_{0,i}, \pi_{0,i}, p_{0,i}, (b/c)_{0,i})/c_{0,i} - v(R_{t,i}, \pi_{t,i}, p_{0,i}, (b/c)_{0,i})/c_{t,i} \tag{49}$$

for each year t and province type i, where $(R_{t,i}, \pi_{t,i}, p_{t,i}, (b/c)_{t,i})$ are the estimated values for year-province-type t, i and where we use 0 to denote the value in the first year of the sample, 1993. The first row of the table reports the mean of $\Delta v_{t,i}$ across provinces, i.e. the total reduction in cost. The second row reports the mean of $\Delta v_{t,i,(p,b)}$ across provinces, i.e. the reduction in cost due to the change in technology. The third row reports the mean of $\Delta v_{t,i,(R,\pi)}$ across provinces, i.e. the reduction in cost due to the disinflation. The fourth row computes the percentage of the total cost due to the changes in technology, by taking the ratio of the entries reported in the second and third rows. Notice that the sum of the second and third rows does not add up to the first row due to the interactions of (p, b) with (R, π).

Figure IV: Theory vs. data (province-year mean): $W/M, n$

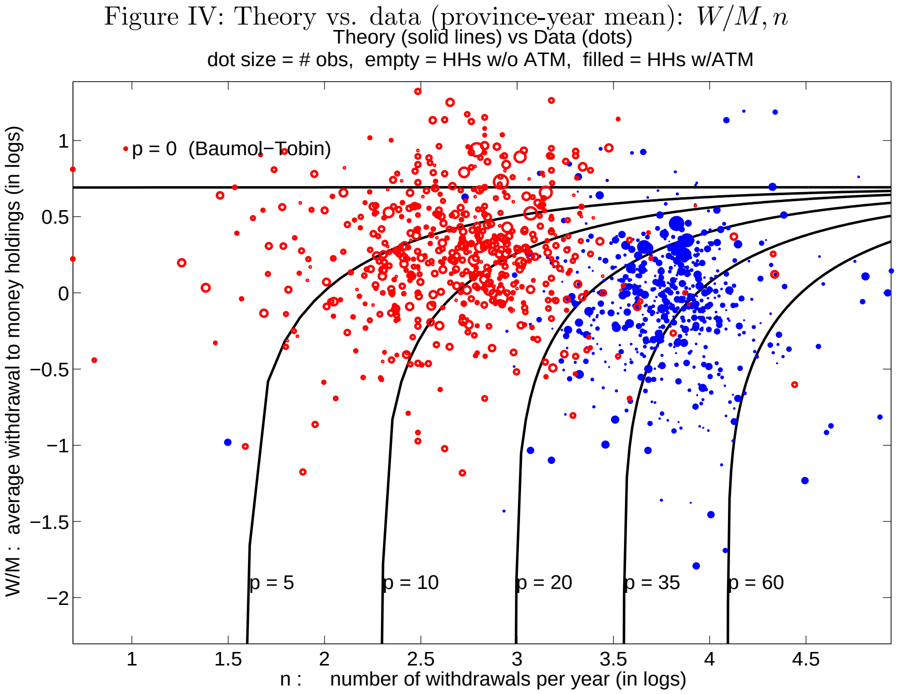

Figure V: Theory vs. data (province-year mean): $\underline{M}/M, n$

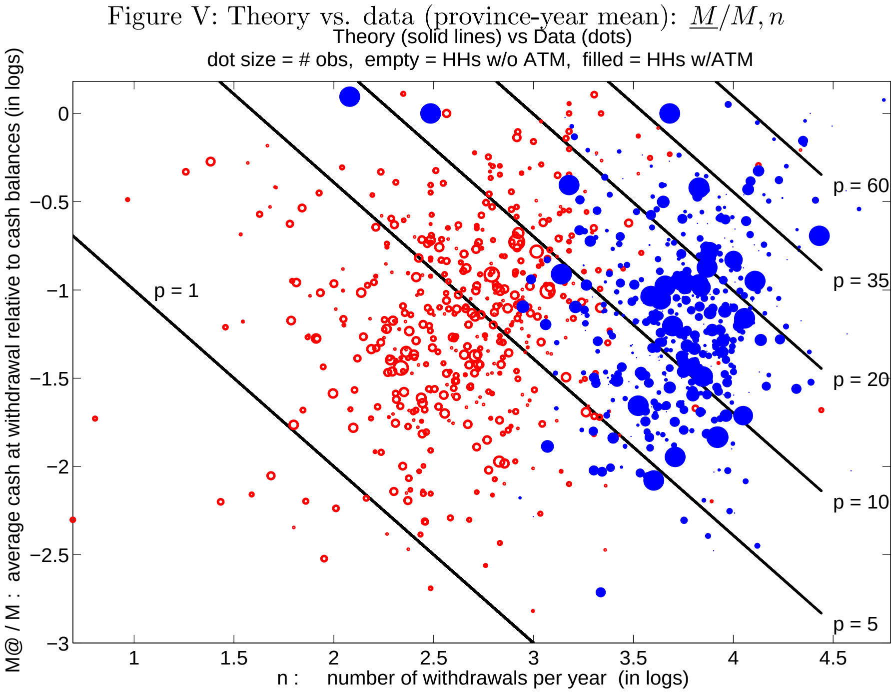

Table XII: Total and counterfactual cumulative reductions in the cost of financing c

Reduction in cost in # cash days for HH w/o ATM						
	1993	1995	1998	2000	2002	2004
Total, due to p, b, R, π	0	0.268	1.28	1.46	1.6	1.55
due to p, b	0	0.102	0.769	0.986	1.07	0.938
due to R, π	0	0.184	0.747	0.846	1.01	1.09
due to p, b, % of total	-	35.6	50.7	53.8	51.4	46.2
Reduction in cost in # cash days for HH w. ATM						
	1993	1995	1998	2000	2002	2004
Total, due to p, b, R, π	0	0.187	0.785	0.87	0.93	0.882
due to p, b	0	0.0842	0.464	0.544	0.56	0.427
due to R, π	0	0.066	0.432	0.499	0.629	0.678
due to p, b, % of total	-	56	51.8	52.1	47.1	38.6

Figure VI: Cost of financing cash purchases (per year)