Understanding Inflation-Indexed Bond Markets: Appendix

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1 Data Construction

The main dataset used in this paper was constructed by downloading data on individual TIPS issues from Bloomberg. The raw data are real clean prices, which can be converted into real yields following exactly the same conventions used to convert nominal prices into nominal yields for nominal bonds. For instance the Matlab function *bndyield* converts the prices of US Treasury bonds into yields and also converts real price sequences into the corresponding real yield sequences.

We used a simple splicing procedure to construct a real yield series as close as possible to constant-maturity 10-year TIPS yields. We first plotted the remaining maturity of outstanding bonds against time, as in Figure A1. Each solid line depicts a single issue, and vertical dashed lines mark the issue of a new bond.

In order to construct a yield series with 10-year maturity, we record the yield series of the bond with the largest maturity less than 10 years until the date when another bond has remaining maturity exactly equal to 10 years, then switch to that new bond. This procedure gives us a 10-year TIPS yield series from January 15, 1998 through December 2008. To verify that the resulting series is smooth across the bond transition dates, we plot the yield series in Figure A2, with the old (recently dropped) bond yields shown in red. Figure A3 shows greater divergence between old and new yield series in the crisis period of the fall of 2008.

We obtain constant-maturity nominal Treasury yields from the CRSP database.

1.1 Nominal TIPS returns

In order to calculate nominal returns on TIPS, we first compute the reference CPI for each TIPS issue. The relevant price index is the "Non-Seasonally Adjusted, All-Urban Consumer Price-Index" (NSA CPI-U). The daily reference CPI is computed as a linear interpolation between the CPI on the first of the month 3 months ago and the CPI of the first of the month 2 months ago according to the formula in Deacon, Derry, and Mirfendereski (2004), p.176. The index ratio for a particular bond is then obtained by dividing the reference CPI by the reference CPI at the issue date.

Note that the relevant issue date here is the date on which a bond was originally intended to be issued rather than the actual issue or settlement. This date usually

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is the 15th just before issuance. For instance the original issue date for the 10-year (2007) indexed note is assumed to be January 15, 1997 rather than the actual settlement date of February 6, 1997 (Deacon, Derry, and Mirfendereski, 2004, p.177).

The US Treasury publishes monthly tables of the daily reference CPI and index ratios on http://www.treasurydirect.gov/instit/annceresult/tipscpi/tipscpi_hiscpi.htm. Thus we can cross-check our index ratio calculations with the published numbers, and have done so for the month of April 2008.

Accrued interest and coupon payments follow the actual/actual daycount convention and assuming that coupon dates always fall on the 15th of a month (Fabozzi 2005, Fabozzi and Mann 2001 Chapter 3). Real accrued interest then is computed as the fraction of a coupon period before settlement times the semiannual coupon payment.

The nominal redemption payment is calculated by multiplying the face value of each bond by $max(\text{indexratio}, 1)$.

Finally, to obtain nominal quantities, we multiply prices, coupon payments and accrued interest by the daily index ratio. The "full" or "dirty" nominal price p_t is obtained by adding accrued interest to the "clean" price. Then we compute daily returns for all days, t, for which price data is also available at day $t-1$ as

$$r_t = \frac{p_t + coupon_t - p_{t-1}}{p_{t-1}}.$$

1.2 Nominal returns on inflation-indexed gilts

Capital and interest values for UK inflation-indexed gilts are adjusted according to the Retail Price Index (RPI). There are in particular two types of inflation-linked gilts outstanding. Prior to the financial year 2005-06 the RPI was incorporated into principal and interest with an 8-month lag. Index-linked gilts issued thereafter follow a 3-month lag methodology.

In order to compute the index ratio for an inflation-linked bond with a 3-month lag, one first needs to compute a daily reference RPI.[2] For example the reference RPI

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[2] The document http://www.dmo.gov.uk/docs//giltsmarket/formulae/igcalc.pdf lays this out in detail.

for 1 June corresponds to the RPI for March, the reference RPI for 1 July corresponds to the RPI for April, etc. For any day other than the first of a month, the reference RPI is obtained from a linear interpolation, according to: Reference RPI for First Day of Coupon Month + ((Day of Coupon Month-1)/Days in Coupon Month) x (Reference RPI for first day of month after coupon month - Reference RPI of first day of coupon month). To obtain a bond's index ratio, one then divides the reference RPI of the settlement date by the reference RPI of the date the bond was first issued. For example, for the $1\frac{1}{4}\%$ Index-linked Treasury Gilt 2017 this is 2/8/2006.[3]

Since November 1998 accrued interest and interest payments for gilts with a 3-month indexation lag have been calculated following the actual/actual day count convention. The formulae for this time period are laid out in UK Debt Management Office (2005). This contains details on how to compute accrued interest for normal, long and short dividends. It also details how to compute accrued interest when a bond goes ex-dividend. The ex-dividend date is 7 business days prior to the coupon date.[4] Comparing with accrued interest calculations available from the DMO, one sees that the nominal dirty price is obtained from the real clean price on date t by multiplying by the index ratio of date t and adding the nominal accrued interest of the subsequent business day.

Since November 1998 dividend payments and accrued interest for gilts with an 8 month indexation lag have been computed according to the actual/actual day count convention. To compute dividend payments and accrued interest, we follow the DMO document or equivalently Deacon, Derry and Mirfendereski (2004). Note that nominal accrued interest for these gilts is obtained by multiplying by the RPI that applies to the next dividend payment, i.e. the RPI of 8 months prior to the next dividend payment.

Prior to 1998, capital and interest were calculated according to the actual/365 day count convention and the corresponding formulae are set out in Bank of England (1997). To compute accrued interest, one does not use intermediate rounding. That is, whenever accrued interest is computed as factor x next dividend payment, one uses the non-rounded value for the next dividend payment.

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[3]The UK Debt Management Office publishes daily index ratios on http://www.dmo.gov.uk/chooseFormat.aspx?rptCode=D10C&page=D10C, and our calculated values have been found to agree with these.

[4]The DMO also provides example calculations online at http://www.dmo.gov.uk/index.aspx?page=Gilts/formulae.

UK Debt Management Office (2005) gives formulae to compute real yields for inflation linked gilts with 3 month and 8 month indexation lags. Yields for 8 month lagged gilts are computed from nominal prices using the reference CPI for the next 6 month and an inflation assumption of 3% thereafter. The formula for 3 month lagged bonds expresses the dirty real price in terms of the real yield. Computations are exactly the same as for nominal yields for nominal bonds, only with nominal quantities replaced by real quantities. The price/yield conversions have been double-checked with the yield/price conversions available from Bloomberg. We computed yields for 8 month lagged bonds by inverting the formula for 3 month lagged inflation indexed bonds. In order to compute a dirty real price, we divide the full dirty price by the 8 month lagged inflation factor. Despite the fact that this inflation factor changes only once a month, inspection of the dirty real price suggests that this price is fairly smooth.

There are some special rules concerning the case when only one dividend period is remaining. In particular, 8 month lagged bonds turn into a nominal bond once the RPI for all future cashflows is known. Since bond data just before maturity tend to be unreliable, we do not use such data.

2   Bond Pricing Model

We assume that the log of the real stochastic discount factor (SDF) $m_{t+1} = \log(M_{t+1})$ follows a linear-quadratic homoskedastic process:

$$-m_{t+1} = x_t + \frac{1}{2}\sigma_m^2 + \varepsilon_{m,t+1}, \tag{1}$$

where x_t follows a conditionally heteroskedastic AR(1) process,

$$x_{t+1} = \mu_x (1 - \phi_x) + \phi_x x_t + v_t \varepsilon_{x,t+1} + \varepsilon'_{x,t+1}, \tag{2}$$

and v_t follows a standard AR(1) process

$$v_{t+1} = \mu_v (1 - \phi_v) + \phi_v v_t + \varepsilon_{v,t+1}, \tag{3}$$

and $\varepsilon_{m,t+1}$, $\varepsilon_{x,t+1}$, $\varepsilon'_{x,t+1}$, and $\varepsilon_{v,t+1}$ are jointly normally distributed zero-mean shocks with constant variance-covariance matrix. We assume that $\varepsilon'_{x,t+1}$ and $\varepsilon_{v,t+1}$ are orthogonal to each other and to the other shocks in the model. We adopt the notation

σ_i^2 to describe the variance of shock ε_i, and σ_{ij} to describe the covariance between shock ε_i and shock ε_j. The conditional volatility of the log SDF (σ_m) describes the price of aggregate market risk or maximum Sharpe ratio in the economy, which we assume to be constant in our model.

The state variable x_t determines the dynamics of the short-term log real interest rate. The price of a single-period zero-coupon real bond satisfies

$$P_{1,t} = E_t \left[\exp \left\{ m_{t+1} \right\} \right], \tag{4}$$

so that its yield $y_{1t} = -\log(P_{1,t})$ equals

$$y_{1,t} = -E_t \left[m_{t+1} \right] - \frac{1}{2} \operatorname{Var}_t \left(m_{t+1} \right) = x_t. \tag{5}$$

Equations (2) and (3) imply that the short-term real interest rate follows a conditionally heteroskedastic AR(1) process. The heteroskedasticity of the real rate is driven by the state variable v_t.

In a standard consumption-based model of the sort we discussed in the previous subsection, v_t would capture time-variation in the dynamics of consumption growth. When v_t is close to zero, shocks to the real interest rate are uncorrelated with the stochastic discount factor, which in a power utility model would imply that shocks to the real rate are uncorrelated with consumption growth. As v_t moves away from zero, the volatility of the real interest rate increases and its covariance with the stochastic discount factor (consumption growth) becomes more positive or more negative. We can interpret v_t as a measure of aggregate uncertainty about long-run growth in the economy. At times where uncertainty about future economic growth increases, real interest rates become more volatile.

The model (1)-(5) implies that the real term structure of interest rates is a linear-quadratic function of the vector of state variables. Specifically, the log price of a n-period zero-coupon real bond, $p_{n,t} = \log(P_{n,t})$, can be written as:

$$p_{n,t} = A_n + B_{x,n} x_t + B_{v,n} v_t + C_{v,n} v_t^2, \tag{6}$$

where the coefficients A_n, $B_{x,n}$, $B_{v,n}$, and $C_{v,n}$ solve a set of recursive equations. These coefficients are functions of the maturity of the bond (n) and the coefficients that determine the stochastic processes for state variables. From equation (5), it is immediate to see that $B_{x,1} = -1$, and the remaining coefficients are zero at $n = 1$.

Equation (6) shows that log inflation-indexed bond prices are linear in the short-term real interest rate x_t, and quadratic in aggregate economic uncertainty v_t. An important property of this model is that bond risk premia are time varying. They are approximately linear in v_t, where the coefficient on v_t is proportional to σ_m^2.

A time varying conditional covariance between the real SDF and the real interest rate implies that the conditional covariance between real bonds and other real assets such as equities also varies over time as a function of v_t. To see this, we now introduce equities into the model. To keep things simple, we assume that the unexpected log return on equities is given by

$$r_{e,t+1} - \mathrm{E}_t\, r_{e,t+1} = \beta_{em}\varepsilon_{m,t+1}. \tag{7}$$

This assumption implies immediately that the covariance between stocks and inflation-indexed bonds is given by

$$\mathrm{Cov}_t(r_{e,t+1}, r_{n,t+1}) = B_{x,n-1}\beta_{em}\sigma_{mx}v_t, \tag{8}$$

which is proportional to v_t. This proportionality is also a reason why we consider two independent shocks to x_t. In the absence of homoskedastic shock $\varepsilon'_{x,t}$ to x_t, our model would imply that the conditional volatility of the short real rate would be proportional to the covariance of stock returns with real bond returns. However, while the two moments appear to be correlated in the data, they are not perfectly correlated.

Combining equation (7) with the standard pricing equation $1 = \mathrm{E}_t[\exp\{m_{t+1} + r_{e,t+1}\}]$ results in the following expression for the equity premium in the model:

$$\mathrm{E}_t\left[r_{e,t+1} - y_{1t}\right] + \mathrm{Var}_t\left(r_{e,t+1} - y_{1t}\right) = \beta_{em}\sigma_m^2. \tag{9}$$

Similarly, equation (7) also implies that the conditional volatility of stock returns is given by

$$\sqrt{\mathrm{Var}_t\left(r_{e,t+1}\right)} = \beta_{em}\sigma_m. \tag{10}$$

Thus the Sharpe ratio on equities is given by σ_m.

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US - Remaining Maturity of Outstanding TIPS against Time

Years

10

8

6

4

2

0

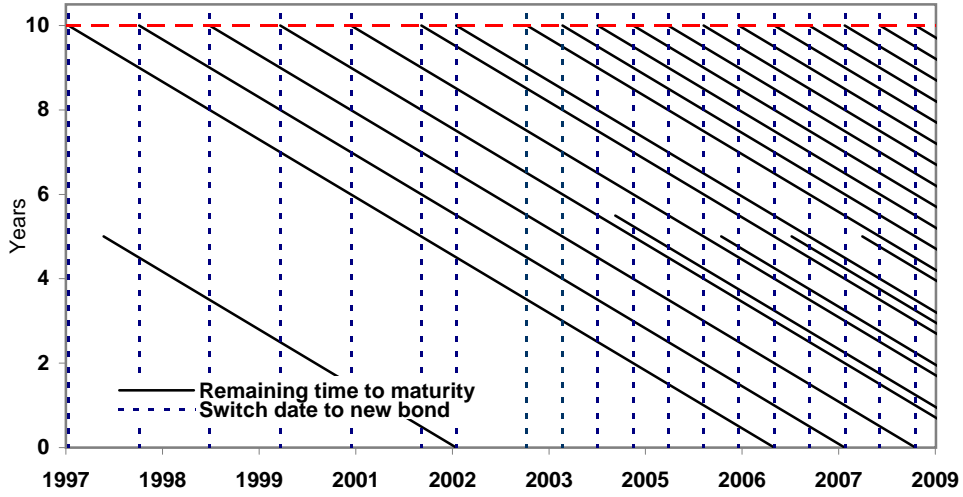1997 1998 1999 2001 2002 2003 2005 2006 2007 2009

—— Remaining time to maturity
---- Switch date to new bond

Figure A1(a)

UK - Remaining Maturity of Outstanding Real Bonds against Time

Years

10

8

6

4

2

0

1991 1993 1996 1999 2001 2004 2007

—— Remaining time to maturity
---- Switch date to new bond
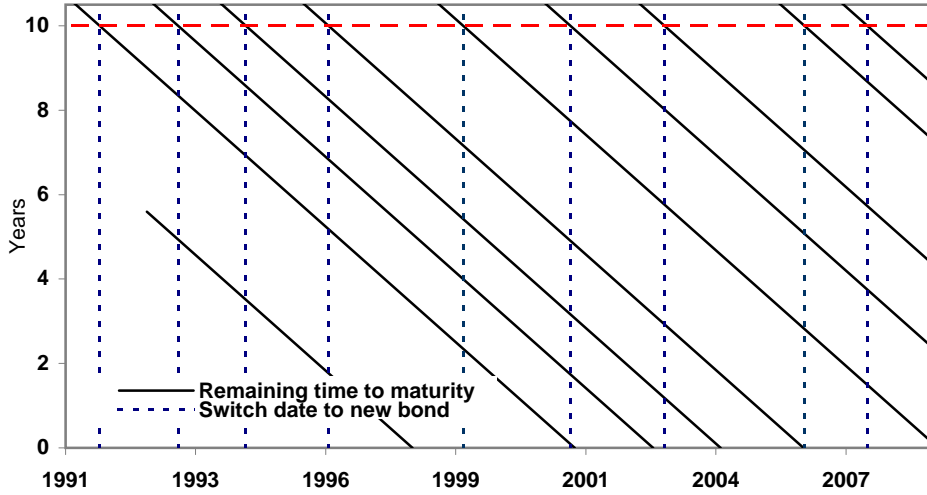
Figure A1(b)

US-10 Year Spliced Yield Series
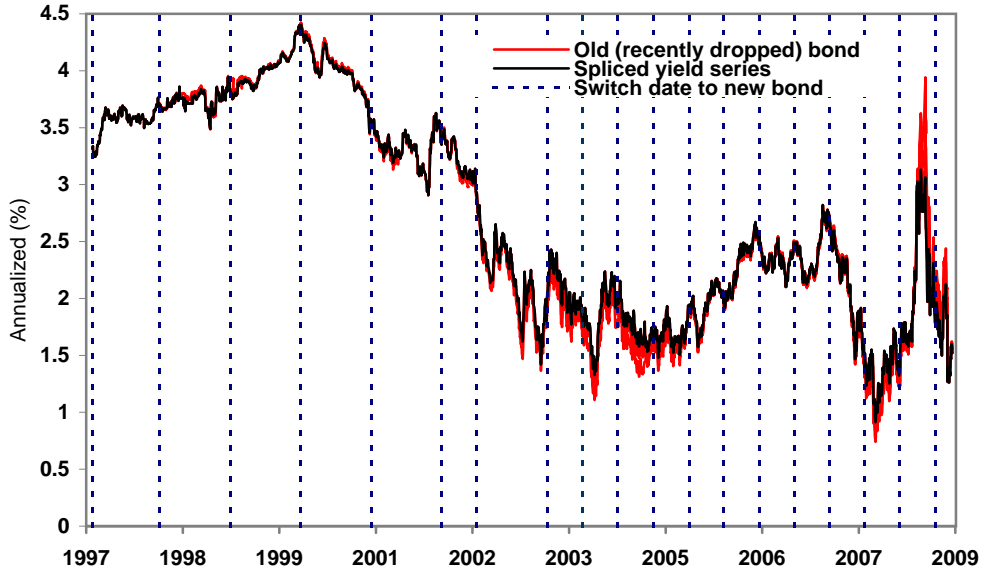
Figure A2(a)

UK-10 Year Spliced Yield Series

Figure A2(b)
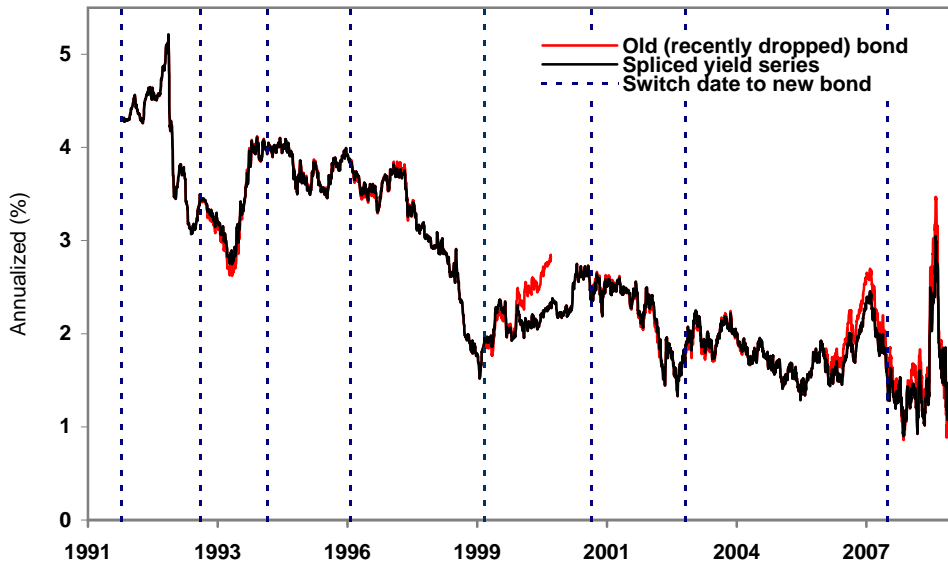
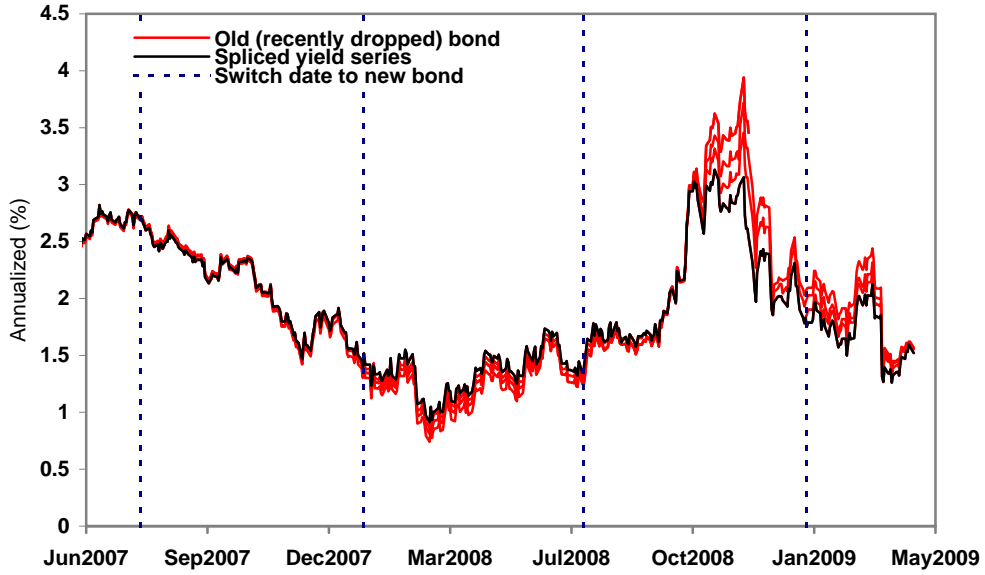US-10 Year Spliced Yield Series Starting June 2007

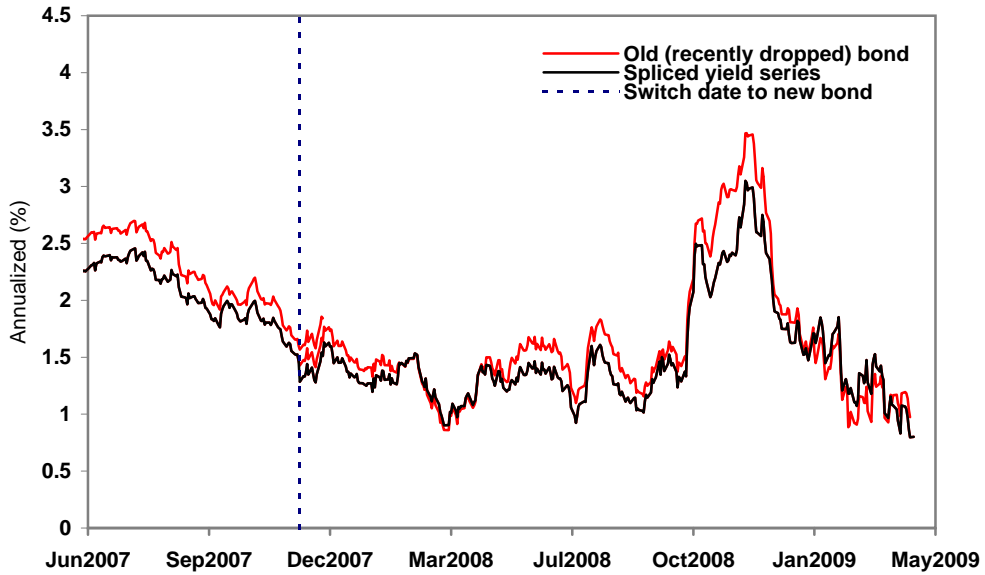**Figure A3(a)**

UK-10 Year Spliced Yield Series Starting June 2007

Figure A3(b)