Technical Appendix for "Optimal Interventions in Markets with Adverse Selection"

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1 Proof of Private Contracts Lemma 1

Let us consider first a separating equilibrium. Ignoring for now the monotonicity constraint, the program of a good bank trying to separate from a bad bank is:

$$\max_{y^l \in [0,y]} E\left[y - y^l | G\right],$$

subject to the break even constraint

$$E\left[y^l | G\right] \geq x - c_0,$$

and the separating constraint

$$E\left[y - y^l | B\right] \leq \tilde{B},$$

where $\tilde{B}$ is the outside option of the bad type. Using the density functions $f(.|\theta)$, we can write the Lagrangian as

$$\mathcal{L} = \int \left(y - y^l\right) f(y|G)\, dy + \lambda \left(\int y^l f(y|G)\, dy - (x - c_0) \right) - \mu \left(\int \left(y - y^l\right) f(y|B)\, dy - \tilde{B} \right),$$

or

$$\mathcal{L} = \int \left(1 - \lambda - \mu \frac{f(y|B)}{f(y|G)}\right) \left(y - y^l\right) f(y|G)\, dy + \lambda \left(E[y] + x - c_0\right) + \mu \tilde{B}.$$

Under A2, $f(.|G)/f(.|B)$ is increasing in y, so $f(.|B)/f(.|G)$ is decreasing, and $1 - \lambda - \mu f(.|B)/f(.|G)$ is increasing. When it is negative, it is optimal to set $y - y^l = 0$. When it turns positive, it is optimal to set $y^l = 0$. This is the well known result of a "live or die" contract. If we know introduce the monotonicity constraint of Inner (1990), it is easy to see that that as long as the contract is strictly increasing, the monotonicity does not bind, and when the contract tends to decrease, the monotonicity constraint forces it to be constant. We therefore obtain a debt contract.

Let us now consider a pooling equilibrium where all banks invest. The program then becomes

$$\max_{y^l \in [0,y]} E\left[y - y^l | G\right],$$

subject to

$$E\left[y^l\right] \geq x - c_0,$$

where $E\left[y^l\right]$ denotes the unconditional expectation of y^l. We can then write the Lagrangian as

$$\mathcal{L} = \int \left(1 - \lambda \frac{f(y)}{f(y|G)}\right) \left(y - y^l\right) f(y|G)\, dy + \lambda \left(E[y] + x - c_0\right).$$

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Again, since f, which is the unconditional distribution of y, is equal to $\pi f(.|G) + (1-\pi) f(.|B)$, we get the "live or die" contract if we do not impose monotonicity, and the debt contract when we impose that y^l be increasing in y. We therefore conclude that deb contract are optimal in all types of equilibria.

2 Proof of Proposition 9

2.1 Equity Injection

In a pure equity injection program, denoted by $\mathcal{E}$, the government injects cash m_α in return for a fraction α of equity. Conditional on investment, the private loan is $l_\alpha = x - c_0 - m_\alpha$ and the private rate is r_α consistent with the pooling equilibrium. The total inside value of equity for type θ conditional on investment is:

$$V(\theta, \mathcal{E}(\theta)) = (1-\alpha) E[y - \min(y, r_\alpha l_\alpha) |\theta].$$

The interest rate r_α is pinned down by l_a via the zero profit condition

$$l_\alpha = E^\pi [\min(y, r_\alpha l_\alpha)]. \tag{1}$$

The initial shareholders retain a fraction $1-\alpha$. The outside option for type θ is $V(\theta, \mathcal{O}(\tilde{r})) = E[y - \min(y, \tilde{r} l_0) |\theta]$ with $l_0 = x - c_0$, and the participation constraint is therefore:

$$V(\theta, \mathcal{E}(\theta)) \geq V(\theta, \mathcal{O}(\tilde{r})). \tag{2}$$

The investment constraint does not depend on α since all shareholders (the government and the old ones) are treated equally. The investment constraint for type θ is:

$$V(\theta, \mathcal{E}(\theta)) \geq (1-\alpha)(E[a|\theta] + c_0 + m_\alpha). \tag{3}$$

The government chooses α and m_α to minimize

$$\Psi_\Pi^{\mathcal{E}} = m_\alpha - \alpha E[y - \min(y, r_\alpha l_\alpha)], \tag{4}$$

subject to participation, investment constraints for all banks and to the interest rate equilibrium condition (1).

As in the proof of Proposition 4 we can show that the participation constraint for the good type implies the participation constraint for the bad type. Moreover, since $r_\alpha(m_a) \leq r_B(m_a)$, the investment constraint for bad banks is always satisfied. Then, the solution depends on whether (2) binds for $\theta = G$, or whether (3) for $\theta = G$ binds.

Suppose first that the investment constraint (3) is slack. Given any cash level m, we can make the participation constraint bind for the good type by choosing α such that

$$\alpha(m) = \frac{\int_0^\infty (\min(y, \tilde{r} l_0) - \min(y, r_\alpha l_\alpha)) f(y|G)\, dy}{E[y - \min(y, r_\alpha l_\alpha) |G]} \tag{5}$$

which must be non-negative. Moreover, we can make the participation constraints for both types of banks to bind by setting $y^l(y, \tilde{r} l_0) = y^l(y, rl)$ which is achieved by a cash level that satisfies $r_\alpha l_\alpha = \tilde{r} l_0$ or $r(x - c_0 - m_\alpha) = \tilde{r}(x - c_0)$, or

$$\hat{m} = \left(1 - \frac{\tilde{r}}{r}\right)(x - c_0). \tag{6}$$

Note that this is an implicit definition since r depends on m through (1). With this cash level we have $\alpha(\hat{m}) = 0$ and the participation constraints for both types of banks bind and the cash injection reaches the lower bound

$$\Psi_\Pi^{\mathcal{E}} = \Psi_\Pi^* = \hat{m}.$$

Note, however, that this only happens when $\alpha = 0$, so it can only be optimal for the government to transfer cash without asking for equity. Moreover, with this cash level we have that $r_\alpha l_\alpha = \tilde{r} l_0$. Assumption A6 only guarantees that $E\left[y-\min\left(y,\tilde{r}l_0\right)|G\right] \geq E\left[a|G\right]+c_0$, so it is clear that the investment constraint (3) may be violated at cash level $\hat{m}$.

When the investment constraint (3) binds for the good type, it pins down m_α. Since this cash level is above the one that sets $\alpha = 0$, we obtain non-negative α from (5). In this case the participation constraint for the bad type is slack and the cost of equity injection does not reach the lower bound. The cost of this program then is

$$\Psi_\Pi^{\mathcal{E}} - \Psi_\Pi^* = (1-\pi)\left(V\left(B,\mathcal{E}\left(B\right)\right) - V\left(B,\mathcal{O}\left(\tilde{r}\right)\right)\right) > 0.$$

2.2 Asset buybacks

The government offers to buy an amount Z of legacy assets for cash m_z. We denote asset buyback programs by $\mathcal{A}$. If a bank opts in the program, the face value of its legacy assets decreases by Z. The payoffs to the government are $y^g = a\frac{Z}{A}$. Define the fraction of buyback by $z \equiv Z/A$. The total value conditional on participation and investment for type θ is

$$V\left(\theta,\mathcal{A}\left(\theta\right)\right) = E\left[y-za-\min\left(y-za,r_z l_z\right)|\theta\right]$$

where $l_z = x - c_0 - m_z$, and where r_z is pinned down by l_z via the zero profit condition

$$l_z = E^\pi\left[\min\left(y-za,r_z l_z\right)\right]. \tag{7}$$

The non-participation payoff for type θ is $V\left(\theta,\mathcal{O}\left(\tilde{r}\right)\right) = E\left[y-\min\left(y,\tilde{r}l_0\right)|\theta\right]$ with $l_0 = x - c_0$. The participation constraint for type θ is

$$V\left(\theta,\mathcal{A}\left(\theta\right)\right) \geq V\left(\theta,\mathcal{O}\left(\tilde{r}\right)\right). \tag{8}$$

The investment constraint is

$$V\left(\theta,\mathcal{A}\left(\theta\right)\right) \geq (1-z)\,E\left[a|\theta\right]+c_0+m_z. \tag{9}$$

The government chooses z and m_z to minimize

$$\Psi_\Pi^{\mathcal{A}} = m_z - zE\left[a\right], \tag{10}$$

subject to participation, investment constraints for all banks and to the interest rate equilibrium condition (7).

As in the proof of Proposition 4 we can show that the participation constraint for the good type, implies the participation constraint for the bad type. Moreover, since $r_z\left(m_z\right) \leq r_B\left(m_z\right)$, the investment constraint for bad banks is always satisfied. Then, the solution depends on whether (8) binds for $\theta = G$, or whether (9) for $\theta = G$ binds.

First we look at the case where the participation constraint for good banks binds, whereas the investment constraints are slack. In this case the government can set $Z = 0$ and $m_z = \hat{m}$ as for the equity injection analyzed above. The program achieves the lower bound on costs

$$\Psi_\Pi^{\mathcal{A}} = \Psi_\Pi^* = \hat{m}.$$

However, this program is not feasible if the investment constraint is violated at $\hat{m}$. In that case, the government sets m_z so that the investment constraint is satisfied with equality. This cash level is higher than $\hat{m}$ and the government chooses a strictly positive Z to make the participation constraint bind for the good type. As in the case of equity, the participation constraint for the bad type is slack and the cost of equity injection does not reach the lower bound:

$$\Psi_\Pi^{\mathcal{A}} - \Psi_\Pi^* = (1-\pi)\left(V\left(B,\mathcal{A}\left(B\right)\right) - V\left(B,\mathcal{O}\left(\tilde{r}\right)\right)\right) > 0.$$

2.3 Comparisons

We want to compare the efficiency of equity injections and asset buybacks. When the outside option is very high the required cash injection is high and the investment constraint is slack. In this case, it is optimal to do a pure cash injection program, and the two interventions are (trivially) equivalent.

The interesting case is when the investment constraint binds. In this case, the government always chooses a non zero value for α and z, and the comparison of the costs $\Psi_{\Pi}^{\mathcal{E}}$ and$\Psi_{\Pi}^{\mathcal{A}}$ is not trivial.

From our previous propositions, we know that since the good type participation constraint always binds, the failure to reach the minimum cost is driven by the gap in the participation constraint of the bad type

$$\begin{aligned}\Psi_{\Pi}^{\mathcal{E}}-\Psi_{\Pi}^{*} &= (1-\pi)\left(V\left(B,\mathcal{E}\left(B\right)\right)-V\left(B,\mathcal{O}\left(\tilde{r}\right)\right)\right) \text{ and}\\ \Psi_{\Pi}^{\mathcal{A}}-\Psi_{\Pi}^{*} &= (1-\pi)\left(V\left(B,\mathcal{A}\left(B\right)\right)-V\left(B,\mathcal{O}\left(\tilde{r}\right)\right)\right),\end{aligned}$$

Then

$$\begin{aligned}\Psi_{\Pi}^{\mathcal{A}} &> \Psi_{\Pi}^{\mathcal{E}} \Leftrightarrow V\left(B,\mathcal{A}\left(B\right)\right)>V\left(B,\mathcal{E}\left(B\right)\right)\\ &\Leftrightarrow E\left[y-za-\min\left(y-za,r_zl_z\right)|B\right]>(1-\alpha)\,E\left[y-\min\left(y,r_\alpha l_\alpha\right)|B\right]\\ &\Leftrightarrow E\left[\min\left(y,r_\alpha l_\alpha\right)-\min\left(y-za,r_zl_z\right)|B\right]>zE\left[a|B\right]-\alpha E\left[y-\min\left(y,r_\alpha l_\alpha\right)|B\right]\\ &\Leftrightarrow \Xi^{\mathcal{E}}\geq\Xi^{\mathcal{A}}\end{aligned}$$

where

$$\begin{aligned}\Xi^{\mathcal{E}} &= (1-\alpha)\,E\left[\min\left(y,r_\alpha l_\alpha\right)|B\right]+\alpha E\left[y|B\right]\\ \Xi^{\mathcal{A}} &= E\left[\min\left(y-za,r_zl_z\right)|B\right]+zE\left[a|B\right]\end{aligned}$$

Let $V\left(B,\mathcal{O}\left(\tilde{r}\right)\right)=V_G^{out}$. When V_G^{out} is high enough we know that $\Psi_{\Pi}^{\mathcal{A}}=\Psi_{\Pi}^{\mathcal{E}}$. We now show that $\Xi^{\mathcal{E}}\geq\Xi^{\mathcal{A}}$ when V_G^{out} goes down, establishing for low values of V_G^{out}, we have that $\Psi_{\Pi}^{\mathcal{A}}>\Psi_{\Pi}^{\mathcal{E}}$, that is equity is cheaper.

For each program, we have 3 equations in 3 unknowns: l, z or α. and $R=rl$. In order to do comparative statics with respect to V_G^{out} we need to totally differentiate a 3×3 system in each case. For the case of equity the system consists of (2), (3) and (1). Similarly for the case of asset buybacks, the system consists of (5), (9) and (7).

For the case of equity, we solve the system and we get that

$$\frac{d\alpha}{dV_G^{out}}=-\frac{1-\alpha}{V\left(G,\mathcal{E}\left(G\right)\right)};\ \frac{dR_\alpha}{dV_G^{out}}=0 \text{ and } \frac{dl_\alpha}{dV_G^{out}}=0.$$

Then, since $\frac{dR_\alpha}{dV_G^{out}}=0$, we get

$$\begin{aligned}\frac{d\Xi^{\mathcal{E}}}{dV_G^{out}} &= \left(E\left[y|B\right]-E\left[\min\left(y,r_\alpha l_\alpha\right)|B\right]\right)\cdot\frac{d\alpha}{dV_G^{out}}\\ &= -E\left[y-\min\left(y,r_\alpha l_\alpha\right)|B\right]\cdot\frac{1-\alpha}{V\left(G,\mathcal{E}\left(G\right)\right)}=-\frac{V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)}.\end{aligned} \tag{11}$$

For asset buyback, we have

$$\begin{aligned}\frac{d\Xi^{\mathcal{A}}}{dV_G^{out}} &= \left(\int_0^A\left(1-F_z\left(a\right)\right)f_a\left(a|B\right)da\right)\frac{dR_z}{dV_G^{out}}+\left(E\left[a|B\right]-\int_0^A aF_z\left(a\right)f_a\left(a|B\right)da\right)\frac{dz}{dV_G^{out}}\\ &= C_2^B\frac{dR_z}{dV_G^{out}}+\left(E\left[a|B\right]-C_1^B\right)\frac{dz}{dV_G^{out}}\end{aligned}$$

where

$$\begin{aligned}C_1^i &= \int_0^A aF_z\left(a\right)f_a\left(a|i\right)da,\ \text{ for } i=B,G,\pi\\ C_2^i &= \int_0^A\left(1-F_z\left(a\right)\right)f_a\left(a|i\right)da,\ \text{ for } i=B,G,\pi\end{aligned} \tag{12}$$

and where

$$\begin{aligned} f_a(a|\pi) &= \pi f_a(a|G) + (1-\pi) f_a(a|G) \\ F_z(a) &= F_v(R_z - (1-z)a) = \begin{cases} 0 \text{ if } a > \frac{R_z}{(1-z)} \\ \leq F_v(R_z) \text{ for } 0 \leq a \leq \frac{R_z}{(1-z)} \end{cases}. \end{aligned} \tag{13}$$

Since

$$\begin{aligned} \frac{dz}{dV_G^{out}} &= \frac{-\left(C_2^G - C_2^\pi\right)}{\left(C_2^G - C_2^\pi\right)\left(E[a|G] - C_1^G\right) + \left(C_1^G - C_1^\pi\right) C_2^G} \\ \frac{dR_z}{dV_G^{out}} &= \frac{C_1^\pi - C_1^G}{\left(C_2^G - C_2^\pi\right)\left(E[a|G] - C_1^G\right) + \left(C_1^G - C_1^\pi\right) C_2^G} \\ \frac{dl_z}{dV_G^{out}} &= \frac{C_1^\pi C_2^G - C_2^\pi C_1^G}{\left(C_2^G - C_2^\pi\right)\left(E[a|G] - C_1^G\right) + \left(C_1^G - C_1^\pi\right) C_2^G} \end{aligned}$$

we get

$$\frac{d\Xi^{\mathcal{A}}}{dV_G^{out}} = -\frac{\left(C_2^G - C_2^\pi\right)\left(E[a|B] - C_1^B\right) + \left(C_1^G - C_1^\pi\right) C_2^B}{\left(C_2^G - C_2^\pi\right)\left(E[a|G] - C_1^G\right) + \left(C_1^G - C_1^\pi\right) C_2^G} \tag{14}$$

We want to show that $\frac{d\Xi^{\mathcal{A}}}{dV_G^{out}} > \frac{d\Xi^{\mathcal{E}}}{dV_G^{out}}$, which from (11) and (14) is equivalent to:

$$-\frac{\left(C_2^G - C_2^\pi\right)\left(E[a|B] - C_1^B\right) + \left(C_1^G - C_1^\pi\right) C_2^B}{\left(C_2^G - C_2^\pi\right)\left(E[a|G] - C_1^G\right) + \left(C_1^G - C_1^\pi\right) C_2^G} > -\frac{V(B, \mathcal{E}(B))}{V(G, \mathcal{E}(G))}$$

or

$$\frac{V(B, \mathcal{E}(B))}{V(G, \mathcal{E}(G))} > \frac{\left(C_2^G - C_2^B\right)\left(E[a|B] - C_1^B\right) + \left(C_1^G - C_1^B\right) C_2^B}{\left(C_2^G - C_2^B\right)\left(E[a|G] - C_1^G\right) + \left(C_1^G - C_1^B\right) C_2^G},$$

where we used the definition of $f_a(a|\pi)$ from (13). This comparison can be shown (details can be found in Subsection (2.6)) to be equivalent to:

$$\frac{E[a|G] - E[a|B]}{E[a|G] - C_1^G + \frac{C_1^G - C_1^B}{C_2^G - C_2^B} C_2^G} > \frac{V(G, \mathcal{E}(G)) - V(B, \mathcal{E}(B))}{V(G, \mathcal{E}(G))} \frac{1-\alpha}{1-\alpha}$$

Now, because of rents

$$\begin{aligned} E[a|G] - E[a|B] &> \frac{V(G, \mathcal{E}(G)) - V(B, \mathcal{E}(B))}{1-\alpha} \\ E[a|G] - E[a|B] &> E[a + v - \min(y, r_\alpha l_\alpha)|G] - E[a + v - \min(y, r_\alpha l_\alpha)|B] \\ 0 &> E[-\min(y, r_\alpha l_\alpha)|B] - E[\min(y, r_\alpha l_\alpha)|G] \end{aligned}$$

which simply says good types repay more. Therefore the numerator is larger on the LHS. It is also clear that $E[a|G] < V(G, \mathcal{E}(G))$ so it is enough to show that

$$C_1^G - \frac{C_1^G - C_1^B}{C_2^G - C_2^B} C_2^G \geq 0.$$

This is equivalent to

$$\begin{aligned} C_1^G\left(C_2^G - C_2^B\right) &\geq C_2^G\left(C_1^G - C_1^B\right) \\ C_2^G C_1^B &\geq C_2^B C_1^G \\ \int_0^A (1 - F_z(a)) f_a(a|G)\, da \int_0^A a F_z(a) f_a(a|B)\, da &\geq \int_0^A (1 - F_z(a)) f_a(a|B)\, da \int_0^A a F_z(a) f_a(a|G)\, da \end{aligned}$$

which using the fact that $\int_0^A f_a(a|G)\,da = \int_0^A f_a(a|B)\,da = 1$, we get that

$$\begin{aligned}
\Longleftrightarrow \quad & \int_0^A aF_z(a)\, f_a(a|B)\,da - \int_0^A aF_z(a)\, f_a(a|G)\,da \qquad (15)\\
\geq \quad & \int_0^A F_z(a)\, f_a(a|G)\,da \int_0^A aF_z(a)\, f_a(a|B)\,da - \int_0^A F_z(a)\, f_a(a|B)\,da \int_0^A aF_z(a)\, f_a(a|G)\,da.
\end{aligned}$$

Now, because $F_z(a)$ (defined in (13)) is decreasing, FOSD implies that

$$\int_0^A F_z(a)\, f_a(a|G)\,da \leq \int_0^A F_z(a)\, f_a(a|B)\,da.$$

Therefore for the RHS of (15) we have that

$$\begin{aligned}
& \int_0^A F_z(a)\, f_a(a|G)\,da \int_0^A aF_z(a)\, f_a(a|B)\,da - \int_0^A F_z(a)\, f_a(a|B)\,da \int_0^A aF_z(a)\, f_a(a|G)\,da \\
\leq \quad & \int_0^A F_z(a)\, f_a(a|B)\,da \left[\int_0^A aF_z(a)\, f_a(a|B)\,da - \int_0^A aF_z(a)\, f_a(a|G)\,da\right],
\end{aligned}$$

and since $F_z(a) < 1$

$$\int_0^A F_z(a)\, f_a(a|B)\,da \left[\int_0^A aF_z(a)\, f_a(a|B)\,da - \int_0^A aF_z(a)\, f_a(a|G)\,da\right] \leq \int_0^A aF_z(a)\, f_a(a|B)\,da - \int_0^A aF_z(a)\, f_a(a|G)\,da$$

which is what we wanted to show.

2.4 Details of How to Obtain $\frac{d\Xi^{\mathcal{E}}}{dV_G^{out}}$

We need to differentiate the system,

$$\begin{aligned}
V(G, \mathcal{E}(G)) &= V_G^{out} \\
V(G, \mathcal{E}(G)) &= (1-\alpha)\left(E[a|G] + x - l_\alpha\right) \\
l_\alpha &= E^\pi\left[\min(y, r_\alpha l_\alpha)\right],
\end{aligned}$$

where

$$\begin{aligned}
V(G, \mathcal{E}(G)) &= (1-\alpha)\, E\left[y - \min(y, r_\alpha l_\alpha)\,|G\right] \\
&= (1-\alpha) \int_0^A \left(\int_{R_\alpha - a}^{\infty} (a + v - R_\alpha)\, f_v(v)\,dv\right) f_a(a|G)\,da,
\end{aligned}$$

where we let $R_\alpha = r_\alpha l_\alpha$. Since

$$d\left[\int_{R_\alpha - a}^{\infty} (a + v - R_\alpha)\, f_v(v)\,dv\right] = -dR_\alpha \int_{R_\alpha - a}^{\infty} f_v(v)\,dv = -dR_\alpha\left(1 - F_\alpha(a)\right)$$

we get

$$dV(G, \mathcal{E}(G)) = -(1-\alpha)\left(\int_0^A \left(1 - F_\alpha(a)\right) f_a(a|G)\,da\right) dR_\alpha - \frac{V(G, \mathcal{E}(G))}{1-\alpha}\,d\alpha \qquad (16)$$

where $F_\alpha(a) = F_v(R_\alpha - a)$.

Similarly

$$E^\pi\left[\min(y, R_\alpha)\right] = \int_0^A \left(\int_0^{R_\alpha - a} y f_v(v)\,dv + \int_{R_\alpha - a}^{\infty} R_\alpha f_v(v)\,dv\right) f_a^\pi(a)\,da$$

where $f_a^{\pi}(a) = \pi f_a(a|G) + (1-\pi) f_a(a|G)$, implies

$$dE[\min(y, R_\alpha)] = dR_\alpha \int_0^A (1 - F_\alpha(a)) f_a^{\pi}(a)\, da \tag{17}$$

Therefore we can write the system

$$\begin{aligned} E^{\pi}[\min(y, r_\alpha l_\alpha)] &= l_\alpha \\ V(G, \mathcal{E}(G)) &= (1-\alpha)(E[a|G] + x - l_\alpha) \\ V(G, \mathcal{E}(G)) &= V_G^{out} \end{aligned}$$

in differential form

$$\begin{aligned} dE^{\pi}[\min(y, r_\alpha l_\alpha)] - dl_\alpha &= 0 \\ dV_{\mathcal{E}}^{in}(G) + (1-\alpha)\, dl_\alpha + \frac{V_{\mathcal{E}}^{in}(G)}{1-\alpha} d\alpha &= 0 \\ dV(G, \mathcal{E}(G)) &= dV_G^{out}. \end{aligned}$$

Using (16) and (17) we get that

$$\begin{aligned} 0 \cdot d\alpha + \int_0^A (1 - F_\alpha(a)) f_a^{\pi}(a)\, da \cdot dR_\alpha + (-1)\, dl_\alpha &= 0 \\ 0 \cdot d\alpha + \left[-(1-\alpha) \left(\int_0^A (1 - F_\alpha(a)) f_a(a|G)\, da \right) \right] \cdot dR_\alpha + (1-\alpha) \cdot dl_\alpha &= 0 \\ -\frac{V_{\mathcal{E}}^{in}(G)}{1-\alpha} \cdot d\alpha + \left[-(1-\alpha) \left(\int_0^A (1 - F_\alpha(a)) f_a(a|G)\, da \right) \right] \cdot dR_\alpha + 0 \cdot dl_\alpha &= dV_G^{out}, \end{aligned}$$

which can be rewritten more compactly as

$$\begin{pmatrix} 0 & B_1 & -1 \\ 0 & B_2 & 1-\alpha \\ B_3 & B_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} d\alpha \\ dR_\alpha \\ dl_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ dV_G^{out} \end{pmatrix}$$

where

$$\begin{aligned} B_1 &= \int_0^A (1 - F_\alpha(a)) f_a^{\pi}(a)\, da \\ B_2 &= -(1-\alpha) \left(\int_0^A (1 - F_\alpha(a)) f_a(a|G)\, da \right) \\ B_3 &= -\frac{V_{\mathcal{E}}^{in}(G)}{1-\alpha}. \end{aligned}$$

Then,

$$\begin{aligned} \det \begin{pmatrix} 0 & B_1 & -1 \\ 0 & B_2 & 1-\alpha \\ B_3 & B_2 & 0 \end{pmatrix} &= -B_1 \det \begin{pmatrix} 0 & 1-\alpha \\ B_3 & 0 \end{pmatrix} - \det \begin{pmatrix} 0 & B_2 \\ B_3 & B_2 \end{pmatrix} \\ &= B_1 B_3 (1-\alpha) + B_2 B_3 \\ &= [B_1(1-\alpha) + B_2] B_3 \end{aligned}$$

We can use Cramer's rule to solve for $d\alpha$, dR_α, dl_α. For that we need the determinants:

$$\det\begin{pmatrix} 0 & B_1 & -1 \\ 0 & B_2 & 1-\alpha \\ dV_G^{out} & B_2 & 0 \end{pmatrix} = -B_1\cdot\det\begin{pmatrix} 0 & 1-\alpha \\ dV_G^{out} & 0 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 0 & B_2 \\ dV_G^{out} & B_2 \end{pmatrix}$$
$$= \left[B_1\left(1-\alpha\right)+B_2\right]\cdot dV_G^{out}$$

$$\det\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1-\alpha \\ B_3 & dV_G^{out} & 0 \end{pmatrix} = 0, \det\begin{pmatrix} 0 & B_1 & 0 \\ 0 & B_2 & 0 \\ B_3 & B_2 & dV_G^{out} \end{pmatrix} = 0$$

Then we get that

$$\begin{aligned} d\alpha &= \frac{\left[B_1\left(1-\alpha\right)+B_2\right]\cdot dV_G^{out}}{\left[B_1\left(1-\alpha\right)+B_2\right]B_3} = \frac{dV_G^{out}}{-\frac{V_{\mathcal{E}}^{in}(G)}{1-\alpha}} \\ &= -\frac{1-\alpha}{V_{\mathcal{E}}^{in}\left(G\right)}\cdot dV_G^{out} \\ dR_\alpha &= 0 \\ dl_\alpha &= 0 \end{aligned}$$

2.5 Details of How to Obtain $\frac{d\Xi^{\mathcal{A}}}{dV_G^{out}}$

We need to differentiate the system,

$$\begin{aligned} V\left(G,\mathcal{A}\left(G\right)\right) &= V_G^{out} \\ V\left(G,\mathcal{A}\left(G\right)\right) &= \left(1-z\right)E\left[a|\theta\right]+x-l_z \\ l_z &= E^\pi\left[\min\left(y,R_z\right)\right], \end{aligned}$$

where $R_z = r_z l_z$,

$$V\left(G,\mathcal{A}\left(G\right)\right) = \int_0^A\left(\int_{R_z-(1-z)a}^{\infty}\left(\left(1-z\right)a+v-R_z\right)f_v\left(v\right)dv\right)f_a\left(a|G\right)da$$

Since $d\left[\int_{R_z-(1-z)a}^{\infty}\left(\left(1-z\right)a+v-r_zl_z\right)f_v\left(v\right)dv\right] = -\int_{R_z-(1-z)a}^{\infty}\left(adz+dR_z\right)f_v\left(v\right)dv$, we get

$$d\left\{V\left(G,\mathcal{A}\left(G\right)\right)\right\} = -\left(E\left[a|G\right]-C_1^G\right)dz - C_2^G dR_z$$

Now:

$$E^\pi\left[\min\left(y-az,R_z\right)\right] = \int_0^A\left(\int_0^{R_z-(1-z)a}\left(\left(1-z\right)a+v\right)f_v\left(v\right)dv+\int_{R_z-(1-z)a}^{\infty}R_zf_v\left(v\right)dv\right)f_a^\pi\left(a\right)da$$

where $f_a^\pi\left(a\right) = \pi f_a\left(a|G\right)+\left(1-\pi\right)f_a\left(a|B\right)$. Then we get that

$$d\left\{E^\pi\left[\min\left(y-az,R_z\right)\right]\right\} = -C_1^\pi dz + C_2^\pi dR_z$$

where the $C's$ and F_z are defined in (12) and (13) respectively.

Then, totally differentiating the system

$$\begin{aligned} l_z - E^\pi\left[\min\left(y-za,r_zl_z\right)\right] &= 0 \\ \left(1-z\right)E\left[a|G\right]+x-l_z-V\left(G,\mathcal{A}\left(G\right)\right) &= 0 \\ V_G^{out}-V\left(G,\mathcal{A}\left(G\right)\right) &= 0 \end{aligned}$$

we get

$$
\begin{aligned}
dl_z - d\left\{E^{\pi}\left[\min\left(y - za, r_z l_z\right)\right]\right\} &= 0 \\
-E\left[a|G\right] dz - dl_z - d\left\{V\left(G, \mathcal{A}\left(G\right)\right)\right\} &= 0 \\
d\left\{V_G^{out}\right\} - d\left\{V\left(G, \mathcal{A}\left(G\right)\right)\right\} &= 0
\end{aligned}
$$

or

$$
\begin{aligned}
C_1^{\pi} dz - C_2^{\pi} + dl_z &= 0 \\
-C_1^G dz + C_2^G dR_z - dl_z &= 0 \\
dV_G^{out} + \left(E\left[a|G\right] - C_1^G\right) dz + C_2^G dR_z &= 0
\end{aligned}
$$

$$
\begin{pmatrix} C_1^{\pi} & -C_2^{\pi} & 1 \\ -C_1^G & C_2^G & -1 \\ E\left[a|G\right] - C_1^G & C_2^G & 0 \end{pmatrix} \cdot \begin{pmatrix} dz \\ dR_z \\ dl_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -dV_G^{out} \end{pmatrix}.
$$

Again, using Cramer's rule we get

$$
\begin{aligned}
\begin{pmatrix} dz \\ dR_z \\ dl_z \end{pmatrix} &= \frac{-dV_G^{out}}{\left(C_2^{\pi} - C_2^G\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^{\pi} - C_1^G\right) C_2^G} \begin{pmatrix} C_2^{\pi} - C_2^G \\ C_1^{\pi} - C_1^G \\ C_1^{\pi} C_2^G - C_2^{\pi} C_1^G \end{pmatrix} \\
&= \frac{dV_G^{out}}{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^{\pi}\right) C_2^G} \begin{pmatrix} C_2^{\pi} - C_2^G \\ C_1^{\pi} - C_1^G \\ C_1^{\pi} C_2^G - C_2^{\pi} C_1^G \end{pmatrix}
\end{aligned}
$$

which means

$$
\begin{aligned}
\frac{dz}{dV_G^{out}} &= \frac{-\left(C_2^G - C_2^{\pi}\right)}{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^{\pi}\right) C_2^G}. \\
\frac{dR_z}{dV_G^{out}} &= \frac{C_1^{\pi} - C_1^G}{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^{\pi}\right) C_2^G}. \\
\frac{dl_z}{dV_G^{out}} &= \frac{C_1^{\pi} C_2^G - C_2^{\pi} C_1^G}{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^{\pi}\right) C_2^G}.
\end{aligned}
$$

2.6 Other Omitted Details

We want to show that

$$
-\frac{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|B\right] - C_1^B\right) + \left(C_1^G - C_1^{\pi}\right) C_2^B}{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^{\pi}\right) C_2^G} > -\frac{V\left(B, \mathcal{E}\left(B\right)\right)}{V\left(G, \mathcal{E}\left(G\right)\right)}
$$

or

$$
\begin{aligned}
\frac{V\left(B, \mathcal{E}\left(B\right)\right)}{V\left(G, \mathcal{E}\left(G\right)\right)} &> \frac{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|B\right] - C_1^B\right) + \left(C_1^G - C_1^{\pi}\right) C_2^B}{\left(C_2^G - C_2^{\pi}\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^{\pi}\right) C_2^G} \text{ or} \\
\frac{V\left(B, \mathcal{E}\left(B\right)\right)}{V\left(G, \mathcal{E}\left(G\right)\right)} &> \frac{\left(C_2^G - C_2^B\right)\left(E\left[a|B\right] - C_1^B\right) + \left(C_1^G - C_1^B\right) C_2^B}{\left(C_2^G - C_2^B\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^B\right) C_2^G}
\end{aligned}
$$

Then

$$
\frac{\left(C_2^G - C_2^B\right)\left(E\left[a|B\right] - C_1^B\right) + \left(C_1^G - C_1^B\right) C_2^B}{\left(C_2^G - C_2^B\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^B\right) C_2^G} - 1 = \frac{\left(C_2^G - C_2^B\right)\left(E\left[a|B\right] - E\left[a|G\right]\right)}{\left(C_2^G - C_2^B\right)\left(E\left[a|G\right] - C_1^G\right) + \left(C_1^G - C_1^B\right) C_2^G}
$$

$$
\begin{aligned}
\frac{\left(C_2^G-C_2^B\right)\left(E\left[a|B\right]-C_1^B\right)+\left(C_1^G-C_1^B\right)C_2^B}{\left(C_2^G-C_2^B\right)\left(E\left[a|G\right]-C_1^G\right)+\left(C_1^G-C_1^B\right)C_2^G} &= 1-\frac{\left(C_2^G-C_2^B\right)\left(E\left[a|G\right]-E\left[a|B\right]\right)}{\left(C_2^G-C_2^B\right)\left(E\left[a|G\right]-C_1^G\right)+\left(C_1^G-C_1^B\right)C_2^G} \\
&= 1-\frac{\left(E\left[a|G\right]-E\left[a|B\right]\right)}{E\left[a|G\right]-C_1^G+\frac{\left(C_1^G-C_1^B\right)}{\left(C_2^G-C_2^B\right)}C_2^G} \\
\frac{V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)}-1 &= \frac{V\left(B,\mathcal{E}\left(B\right)\right)-V\left(G,\mathcal{E}\left(G\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)} \\
\frac{V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)} &= 1-\frac{V\left(G,\mathcal{E}\left(G\right)\right)-V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)}
\end{aligned}
$$

Then

$$
\frac{V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)}>\frac{\left(C_2^G-C_2^B\right)\left(E\left[a|B\right]-C_1^B\right)+\left(C_1^G-C_1^B\right)C_2^B}{\left(C_2^G-C_2^B\right)\left(E\left[a|G\right]-C_1^G\right)+\left(C_1^G-C_1^B\right)C_2^G}
$$

is equivalent to

$$
1-\frac{V\left(G,\mathcal{E}\left(G\right)\right)-V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)}>1-\frac{\left(E\left[a|G\right]-E\left[a|B\right]\right)}{E\left[a|G\right]-C_1^G+\frac{\left(C_1^G-C_1^B\right)}{\left(C_2^G-C_2^B\right)}C_2^G}
$$

or

$$
\frac{E\left[a|G\right]-E\left[a|B\right]}{E\left[a|G\right]-C_1^G+\frac{C_1^G-C_1^B}{C_2^G-C_2^B}C_2^G}>\frac{V\left(G,\mathcal{E}\left(G\right)\right)-V\left(B,\mathcal{E}\left(B\right)\right)}{V\left(G,\mathcal{E}\left(G\right)\right)}.
$$

3 Optimal Menus: Missing Parts of Proof of Proposition 10

3.1 Asset-Buyback Menus

Again, the revelation principle implies that without loss we can assume that each program consists of an option for good banks and an option for bad banks:

$$
m_G, z_G \text{ and } m_B, z_B.
$$

The government offers cash m against z assets. Then, the participation payoff for type θ bank is $V\left(\theta,\mathcal{P}_\theta\right)=E\left[y-y^l\left(y,r_\theta l_{m_\theta}\right)-y^g\left(a,z_\theta\right)|\theta\right]$, where $l_{m_\theta}=x-c_0-m_\theta$, whereas the non-participation payoff for type θ is $V\left(\theta,\mathcal{O}\left(\tilde{r}\right)\right)=E\left[y-y^l\left(y,\tilde{r}l_0\right)|\theta\right]$, where $l_0=x-c_0$.

The constraints are

$$
\begin{aligned}
IC_B &: \; E\left[a+v-y^l\left(y,r_Bl_{m_B}\right)-y^g\left(a,z_B\right)|B\right]\geq E\left[a+v-y^l\left(y,r_Gl_{m_G}\right)-y^g\left(a,z_G\right)|B\right] \\
IC_G &: \; E\left[a+v-y^l\left(y,r_Gl_{m_G}\right)-y^g\left(a,z_G\right)|G\right]\geq E\left[a+v-y^l\left(y,r_Bl_{m_B}\right)-y^g\left(a,z_B\right)|G\right] \\
PC_G &: \; E\left[a+v-y^l\left(y,r_Gl_{m_G}\right)-y^g\left(a,z_G\right)|G\right]\geq E\left[y-y^l\left(y,\tilde{r}l_0\right)|G\right] \\
PC_B &: \; E\left[a+v-y^l\left(y,r_Bl_{m_B}\right)-y^g\left(a,z_B\right)|B\right]\geq E\left[y-y^l\left(y,\tilde{r}l_0\right)|B\right] \\
z_G &: \; z_G\geq 0 \\
z_B &: \; z_B\geq 0
\end{aligned}
$$

From the IC we get that

$$
\begin{aligned}
E\left[\left(y^l\left(y,r_Bl_{m_B}\right)+y^g\left(a,z_B\right)\right)|G\right] &\geq E\left[y^l\left(y,r_Gl_{m_G}\right)+y^g\left(a,z_G\right)|G\right] \\
E\left[y^l\left(y,r_Gl_{m_G}\right)+y^g\left(a,z_G\right)|B\right] &\geq E\left[\left(y^l\left(y,r_Bl_{m_B}\right)+y^g\left(a,z_B\right)\right)|B\right].
\end{aligned}
$$

Consider the following menu: $z_G=z_B=0$ and m_G and m_B be such that $y^l\left(y,r_Gl_{m_G}\right)=y^l\left(y,r_Bl_{m_B}\right)=y^l\left(y,\tilde{r}l_0\right)$. Notice again, that in this menu $m_G\neq m_B$ which is necessary in order to achieve separation. With such a menu we have the incentive and the participation constraints of both types hold with equality. Hence this menu is feasible and it achieves the minimal cost for the government. Notice also that in deriving this program we have assumed that good banks invest when they choose the option for the bad banks. This is indeed the case, which follows from the fact that $y^l\left(y,r_Bl_{m_B}\right)=y^l\left(y,\tilde{r}l_0\right)$ and from Assumption A6.

3.2 Debt Guarantee Menus

Now we turn to examine debt guarantee programs. Here the two menus are

$$\phi_G, S_G \text{ and } \phi_B, S_B.$$

In such a program the government guarantees new loans up to S_θ in exchange for a fee $\phi_\theta S_\theta$. Given such a program we have that

$$y^u\left(y, r_\theta l^u\right) = \min(y, r_\theta l^u),\ \ y^s\left(y, S\right) = \min(y - y^u, S) \text{ and } y^g = 0.$$

Then, the participation payoff for type θ bank is $V\left(\theta, \mathcal{P}_\theta\right) = E\left[y - y^u\left(y, r_\theta l_\theta^u\right) - y^s\left(y, S_\theta\right)|\theta\right]$, where $l_\theta^u = l_0 - (1-\phi_\theta)\, S_\theta$, whereas the non-participation payoff for type θ is $V\left(\theta, \mathcal{O}\left(\tilde{r}\right)\right) = E\left[y - y^l\left(y, \tilde{r}l_0\right)|\theta\right]$, where $l_0 = x - c_0$.

The constraints are

$$\begin{aligned}
IC_B &: E\left[a + v - y^u\left(y, r_B l^u\right) - y^s\left(y, S_B\right)|B\right] \geq E\left[a + v - y^u\left(y, r_G l^u\right) - y^s\left(y, S_G\right)|B\right] \\
IC_G &: E\left[a + v - y^u\left(y, r_G l^u\right) - y^s\left(y, S_G\right)|G\right] \geq E\left[a + v - y^u\left(y, r_B l^u\right) - y^s\left(y, S_B\right)|G\right] \\
PC_G &: E\left[a + v - y^u\left(y, r_G l^u\right) - y^s\left(y, S_G\right)|G\right] \geq E\left[y - y^l\left(y, \tilde{r}l_0\right)|G\right] \\
PC_B &: E\left[a + v - y^u\left(y, r_B l^u\right) - y^s\left(y, S_B\right)|B\right] \geq E\left[y - y^l\left(y, \tilde{r}l_0\right)|B\right]
\end{aligned}$$

Combining the two incentive constraints we get that

$$\begin{aligned}
E\left[y^u\left(y, r_G l^u\right) + y^s\left(y, S_G\right)|B\right] &\geq E\left[y^u\left(y, r_B l^u\right) + y^s\left(y, S_B\right)|B\right] \\
E\left[y^u\left(y, r_B l^u\right) + y^s\left(y, S_B\right)|G\right] &\geq E\left[y^u\left(y, r_G l^u\right) + y^s\left(y, S_G\right)|G\right].
\end{aligned}$$

Consider a program where $y^u\left(y, r_B l^u\right) + y^s\left(y, S_B\right) = y^u\left(y, r_G l^u\right) + y^s\left(y, S_G\right) = y^l\left(y, \tilde{r}l_0\right)$. This program is feasible as it makes all constraints hold with equality, which, in turn, also implies that it is optimal. Moreover, good banks do want to invest when choosing the option for bad banks, exactly for the same reasons we explained in the case of asset buybacks.