TECHNICAL APPENDIX – NOT FOR PUBLICATION

6 Additional results

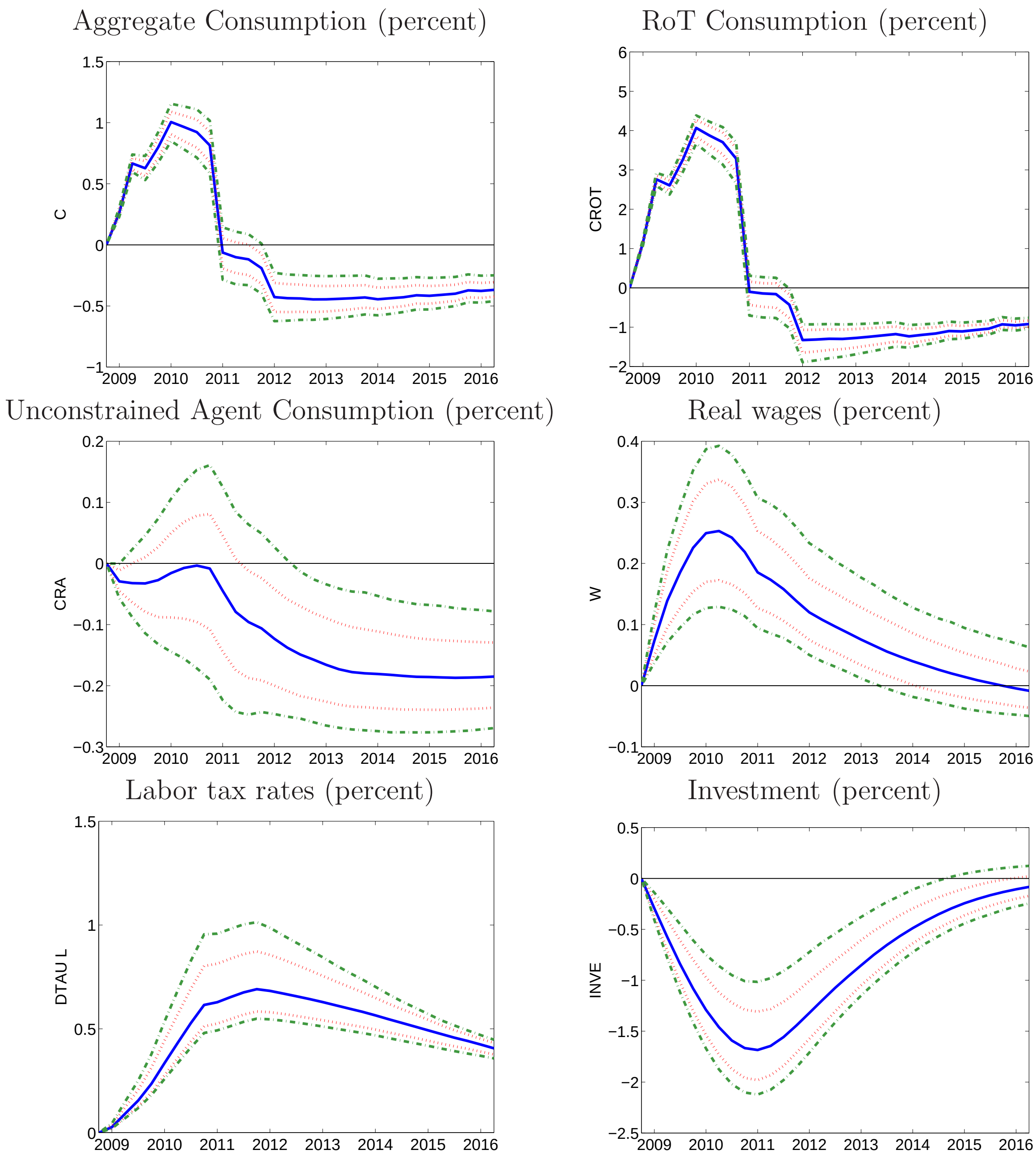

Figure 15: Benchmark impact of ARRA: Consumption, Investment, Tax rates, and Real Wages.

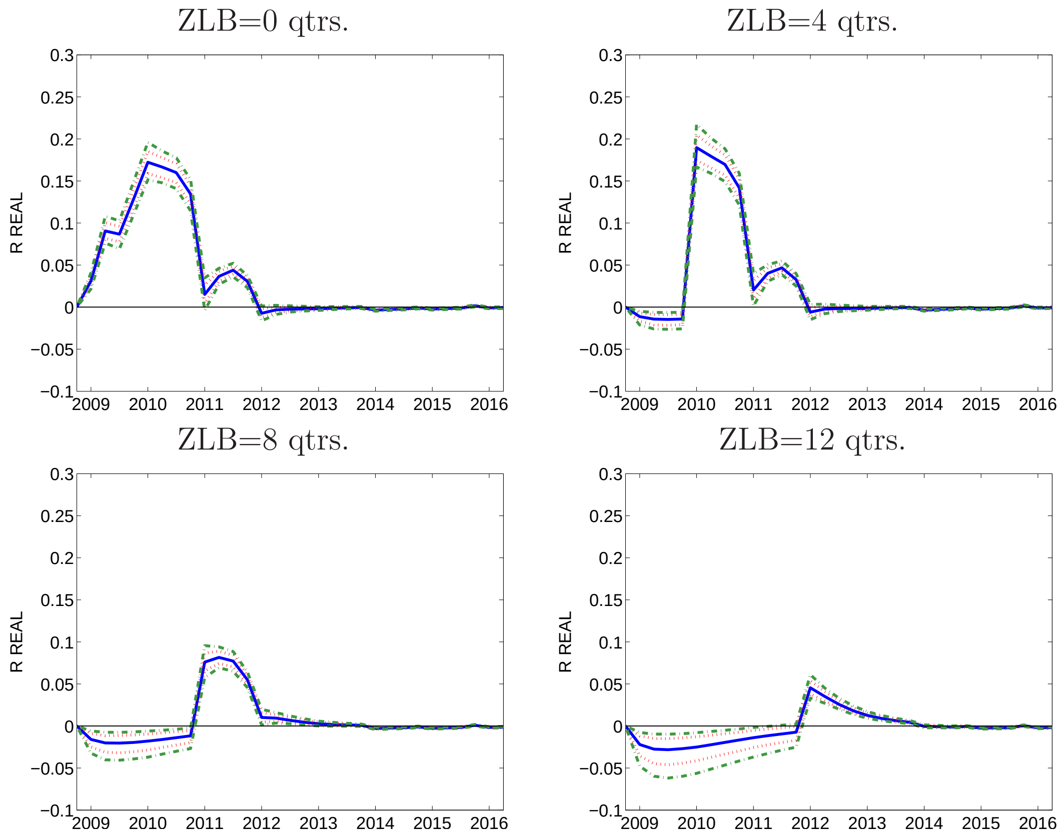

Figure 16: Impact of ARRA on real interest rates for varying ZLB length.

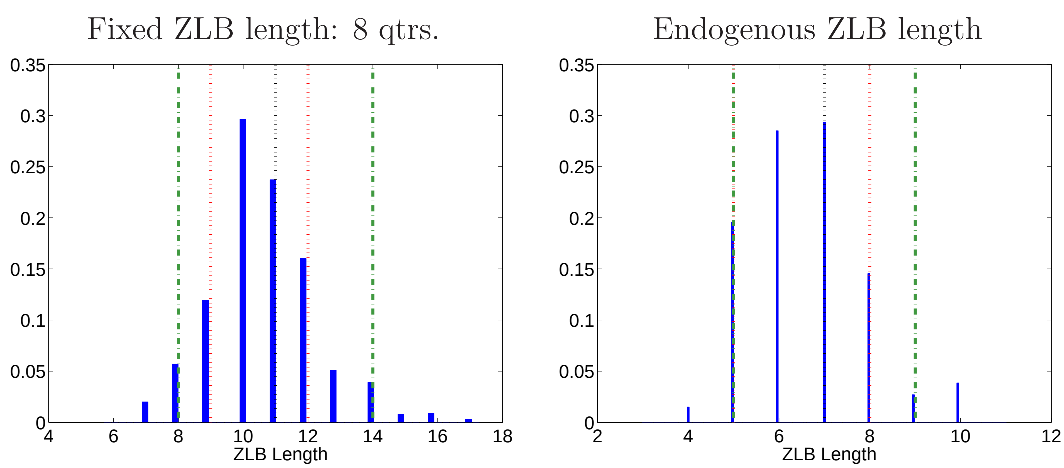

Figure 17: *ZLB duration implied by Taylor rule.*

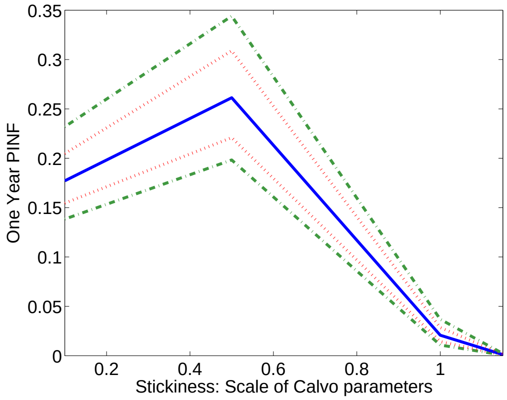

Figure 18: *Inflation response: sensitivity to price and wage stickiness.*

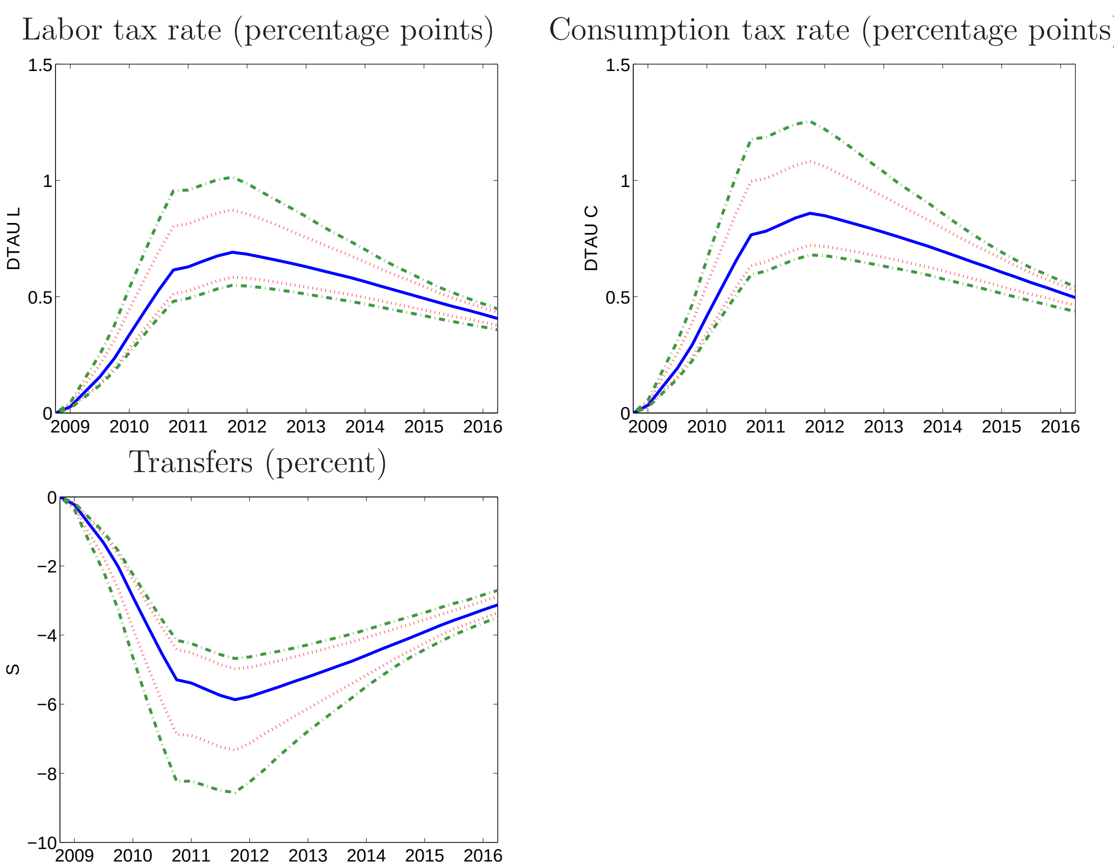

Figure 19: Changes in tax rates and lump-sum transfers due to stimulus.

7 Categorizing stimulus spending

Table 9: Categorizing the stimulus – Government Consumption

Item	Amount (bn USD)	Share
Dept. of Defense	4.53	0.59
Employment and Training	4.31	0.56
Legislative Branch	0.03	0
National Coordinator for Health Information Technology	1.98	0.26
National Institute of Health	9.74	1.26
Other Agriculture, Food, FDA	3.94	0.51
Other Commerce, Justice, Science	5.36	0.69
Other Dpt. of Education	2.12	0.28
Other Dpt. of Health and Human Services	9.81	1.27
Other Financial Services and gen. Govt	1.31	0.17
Other Interior and Environment	4.76	0.62
Special education	12.2	1.58
State and local law enforcement	2.77	0.36
State Fiscal Relief	90.04	11.68
State fiscal stabilization fund	53.6	6.95
State, foreign operations, and related programs	0.6	0.08
Other	2.55	0.33
Consumption	209.64	27.2

Table 10: Categorizing the stimulus – Government Investment

Item	Amount (bn USD)	Share
Broadband Technology opportunities program	4.7	0.61
Clean Water and Drinking Water State Revolving Fund	5.79	0.75
Corps of Engineers	4.6	0.6
Distance Learning, Telemedicine, and Broadband Program	1.93	0.25
Energy Efficiency and Renewable Energy	16.7	2.17
Federal Buildings Fund	5.4	0.7
Health Information Technology	17.56	2.28
Highway construction	27.5	3.57
Innovative Technology Loan Guarantee	6	0.78
NSF	2.99	0.39
Other Energy	22.38	2.9
Other transportation	20.56	2.67
Investment	136.09	17.66

Table 11: Categorizing the stimulus – Transfers

Item	Amount (bn USD)	Share
Assistance for the unemployed	0.88	0.11
Economic Recovery Programs, TANF, Child support	18.04	2.34
Health Insurance Assistance	25.07	3.25
Health Insurance Assistance	-0.39	-0.05
Low Income Housing Program	0.14	0.02
Military Construction and Veteran Affairs	4.25	0.55
Other housing assistance	9	1.17
Other Tax Provisions	4.81	0.62
Public housing capital fund	4	0.52
Refundable Tax Credits	68.96	8.95
Student financial assistance	16.56	2.15
Supplemental Nutrition Assistance Program	19.99	2.59
Tax Provisions	214.56	27.84
Unemployment Compensation	39.23	5.09
Transfers and Tax cuts	425.09	55.15

8 Backing out the unemployment rate

To back out the model implications for the unemployment rate, we regress the time series for hours worked used for the model estimation on the average quarterly unemployment rate. Table 12 shows the regression results. Figure 20 displays the actual and fitted unemployment rate. Multiplying hours worked on the OLS regression coefficient gives the implied change in the unemployment rate.

Table 12: *OLS regression estimates of unemployment rate on the model-implied employment measure.*

	Constant	Employment (lab_t)	R^2
Unemployment Rate (UR_t)	5.60	-0.46	0.77
	(5.51, 5.69)	(-0.49, -0.43)	

Sample period: 1948:1 – 2008:4. Unemployment rate is the arithmetic mean over the quarter. 95 percent confidence intervals in parentheses. Labor input in the model is measured as $\text{lab}_t \equiv \log \frac{\text{Avg. hours}_t \times \text{Employment}_t}{\text{Population}_t} - \text{mean}$. 95 percent OLS confidence intervals in parentheses.

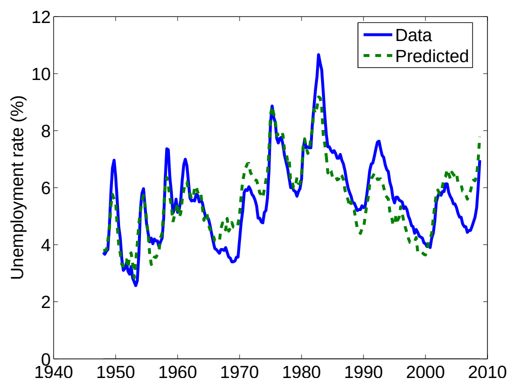

Figure 20: *Regression of quarterly unemployment rate on the model-implied employment measure: Actual vs. predicted unemployment rate.*

9 Model Appendix

Apart from the model extensions due to the introduction of government capital, rule of thumb consumers, and distortionary taxation, the following model appendix follows mostly the appendix of Smets and Wouters (2007), with minor changes to unify the notation.

9.1 Production

Final goods are produced in a competitive final goods sector which uses differentiated intermediate inputs, supplied by monopolistic intermediate producers.

9.1.1 Final goods producers

The representative final goods producer maximizes profits by choosing intermediate inputs $Y_t(i), i \in [0,1]$, subject to a production technology which generalizes a CES production function: Objective:

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di \quad \text{s.t.} \quad \int_0^1 G\left(\frac{Y_t(i)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right) di = 1. \tag{9.1}$$

$G(\cdot)$ is the **?** aggregator, which generalizes CES demand by allowing the elasticity of demand to increase with relative prices: $G' > 0, \quad G'' < 0, \quad G(1; \tilde{\epsilon}_t^{\lambda,p}) = 1$. $\tilde{\epsilon}_t^{\lambda,p}$ is a shock to the production technology which changes the elasticity of substitution.

Denote the Lagrange multiplier on the constraint by Ξ_t^f. If a positive solution to equation (9.1) exists it satisfies the following conditions

$$[Y_t] \quad P_t = \Xi_t^f \frac{1}{Y_t} \int_0^1 G'\left(\frac{Y_t(i)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right) \frac{Y_t(i)}{Y_t} di,$$

$$[Y_t(i)] \quad P_t(i) = \Xi_t^f \frac{1}{Y_t} G'\left(\frac{Y_t(i)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right).$$

From these two equations, we obtain an expression for the aggregate price index and intermediate inputs. The price index is given by:

$$P_t = \int_0^1 \frac{Y_t(i)}{Y_t} P_t(i) di. \tag{9.2}$$

Solving for intermediate input demands:

$$Y_t(i) = Y_t G'^{-1}\left(\frac{P_t(i)Y_t}{\Xi_t^f}\right) = Y_t G'^{-1}\left(\frac{P_t(i)}{P_t}\int_0^1 G'\left(\frac{Y_t(j)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right)\frac{Y_t(j)}{Y_t}dj\right). \tag{9.3}$$

For future reference, note that the relative demand curves $\mathrm{y}_t(i) \equiv \frac{Y_t(i)}{Y_t}$ are downward-sloping in the relative price $\frac{P_t(i)}{P_t}$ with an decreasing elasticity as the relative quantity increases. For simplicity, the dependence of the $G(\cdot)$ aggregator on the shock $\tilde{\epsilon}_t^{\lambda,p}$ is suppressed:

$$\eta_p(\mathrm{y}_t(i)) \equiv -\frac{P_t(i)}{Y_t(i)}\frac{d\mathrm{y}_t(i)}{dP_t(i)}\Big|_{dY_t = d\Xi_t^f = 0} = -\frac{G'(\mathrm{y}_t(i))}{\mathrm{y}_t(i)G''(\mathrm{y}_t(i))} \tag{9.4}$$

$$\begin{aligned}
\hat{\eta}_p(\mathrm{y}_t(i)) &\equiv \frac{P_t(i)}{\eta_p(\mathrm{y}_t(i))}\frac{d\eta_p(\mathrm{y}_t(i))}{dP_t(i)} = 1 + \eta_p + \eta_p\frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i) \\
&= 1 + \eta_p(\mathrm{y}_t(i))\left(2 + \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i) - 1\right) \\
&= 1 + \eta_p(\mathrm{y}_t(i))\left(\frac{2 + \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i)}{1 - \eta_p(\mathrm{y}_t(i))^{-1}}(1 - \eta_p(\mathrm{y}_t(i))^{-1}) - 1\right) \\
&\equiv 1 + \frac{1 + \lambda^p(\mathrm{y}_t(i))}{\lambda^p(\mathrm{y}_t(i))}\left(\frac{1}{[1 + \lambda^p(\mathrm{y}_t(i))]A_p(\mathrm{y}_t(i))} - 1\right),
\end{aligned} \tag{9.5}$$

where the last line defines the mark-up $\lambda_t^p(\mathrm{y}_t(i)) \equiv \frac{1}{\eta_p(\mathrm{y}_t(i)) - 1}$ and $A_p(\mathrm{y}_t(i)) \equiv \frac{\lambda^p(\mathrm{y}_t(i))}{2 + \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i)}$. The model will be parameterized in terms of $\hat{\epsilon}(1)$, the change in the own price elasticity of demand along the balanced growth path. To that end, it is convenient to solve for A_p in terms of the mark-up and the $\hat{\epsilon}$:

$$A_p(\mathrm{y}) = \frac{1}{\lambda^p(\mathrm{y})\hat{\eta}_p(\mathrm{y}) + 1}. \tag{9.6}$$

Finally, note that in the Dixit-Stiglitz case $G(\mathrm{y}) = \mathrm{y}^{\frac{1}{1+\lambda^p}}$ so that the elasticity of demand is constant at $\eta_p(\mathrm{y}) = \frac{1}{\lambda^p} + 1 \forall \mathrm{y}$ and consequently $\hat{\eta}_p = 0$.

9.1.2 Intermediate goods producers

There is a unit mass of intermediate producers, indexed by $i \in [0,1]$. Each producer is the monopolistic supplier of good i. They rent capital services K_t^{eff} and hire labor n_t to maximize profits intertemporally, taking as given rental rates R_t^k and wages W_t. Given a Calvo-style pricing friction, their profit-maximization problem is dynamic.

Production is subject to a fixed cost and the gross product is produced using a Cobb-Douglas technology at the firm level. Government capital K_t^g increases total factor productivity in each firm, but is subject to a congestion effect as overall production increases, similar to the congestion effects in the AK model in **?**. Firms fail to internalize the effect of their decisions on public sector productivity. Net output is therefore given by:

$$Y_t(i) = \tilde{\epsilon}_t^a \left(\frac{K_{t-1}^g}{\int_0^1 Y_t(j)dj + \Phi \mu^t} \right)^{\frac{\zeta}{1-\zeta}} K_t^{eff}(i)^{\alpha} [\mu^t n_t(i)]^{1-\alpha} - \mu^t \Phi, \tag{9.7}$$

where $\Phi\mu^t$ represent fixed costs which grow at the rate of labor augmenting technical progress and $K_t(i)^{eff}$ denotes the capital services rented by firm i. $\tilde{\epsilon}_t^a$ denotes a stationary TFP process.

To see the implications of the congestion costs, consider the symmetric case that $Y_t(i) = Y_t, K_t^{eff}(i) = K_t^{eff} \forall i$, which is the case along the symmetric balanced growth path and in the flexible economy. We then obtain the following aggregate production function:

$$Y_t = \epsilon_t^a {K_{t-1}^g}^{\zeta} {K_t^{eff}}^{\alpha(1-\zeta)} [\mu^t n_t]^{(1-\alpha)(1-\zeta)} - \mu^t \Phi, \quad \epsilon_t^a \equiv (\tilde{\epsilon}_t^a)^{1-\zeta}. \tag{9.8}$$

Choose units such that $\bar{\epsilon}^a \equiv 1$.

To solve a firm's profit maximization problem, note that it is equivalent to minimizing costs (conditional on operating) and then choosing the quantity optimally. Consider the cost-minimization problem first:

$$\min_{K_t(i),n_t(i)} W_t n_t(i) + R_t^k K_t(i) \text{ s.t. } (9.7).$$

Denote the Lagrange multiplier on the production function by MC_t – producing a marginal unit more raises costs marginally by MC_t. The static FOC are necessary and sufficient, given $Y_t(i)$:

$$\begin{aligned}
[n_t(i)] \quad & MC_t(i)(1-\alpha)\frac{Y_t(i)+\mu^t\Phi}{n_t(i)} = W_t, \\
[K_t(i)] \quad & MC_t(i)\alpha\frac{Y_t(i)+\mu^t\Phi}{K_t(i)} = R_t^k.
\end{aligned}$$

The FOC can be used to solve for the optimal capital-labor ratio in production and marginal costs:

$$\frac{k_t(i)}{n_t(i)} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}, \tag{9.9}$$

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{W_t^{1-\alpha}(R_t^k)^\alpha\mu^{-(1-\alpha)t}}{\left(\frac{K_{t-1}^g}{Y_t+\mu^t\Phi}\right)^{\frac{\zeta}{1-\zeta}}\epsilon_t^a}, \tag{9.10}$$

$$mc_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{w_t^{1-\alpha}(r_t^k)^\alpha}{\left(\frac{\mu k_{t-1}^g}{y_t+\Phi}\right)^{\frac{\zeta}{1-\zeta}}\epsilon_t^a},$$

$$mc_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{w_t^{1-\alpha}(r_t^k)^\alpha}{\left(\frac{\mu k_{t-1}^g}{y_t+\Phi}\right)^{\frac{\zeta}{1-\zeta}}\epsilon_t^a}, \tag{9.11}$$

where lower case letters denote detrended, real variables as applicable:

$$k_t \equiv K_t\mu^{-t}, y_t \equiv Y_t\mu^{-t}, w_t \equiv \frac{W_t}{\mu^t P_t}, r_t^k \equiv \frac{R_t^k}{P_t}, mc_t \equiv \frac{MC_t}{P_t}.$$

For future reference, it is useful to detrend the FOC:

$$w_t = mc_t(i)(1-\alpha)\frac{y_t(i)+\Phi}{n_t(i)}, \tag{9.12a}$$

$$r_t^k = mc_t(i)\alpha \frac{y_t(i) + \Phi}{k_t(i)}. \tag{9.12b}$$

Given the solution to the static cost-minimization problem, the firm maximizes the present discounted value of its profits by choosing quantities optimally, taking as given its demand function (9.3), the marginal costs of production (9.10), and respecting the Calvo-style price setting friction. The Calvo-friction implies that a firm can re-set its price in each period with probability $1 - \zeta_p$ and otherwise indexes its price to an average of current and past inflation $\prod_{l=1}^{s} \pi_{t+l-1}^{\iota_p} \bar{\pi}^{1-\iota_p}$. In each period t that the firm can change its prices it chooses:

$$P_t^*(i) = \arg\max_{\tilde{P}_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s} P_t}{\xi_t P_{t+s}} \Big[\tilde{P}_t(i) \Big(\prod_{l=1}^{s} \pi_{t+l-1}^{\iota_p} \bar{\pi}^{1-\iota_p}\Big) - MC_{t+s}(i)\Big] Y_{t+s}(i),$$

subject to (9.3) and (9.10). $\frac{\bar{\beta}^s \xi_{t+s}}{\xi_t}$ denotes the (non-credit constrained) representative household's stochastic discount factor and $\pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes period t inflation.

To solve the problem, it is useful to define $\chi_{t,t+s}$ such that in the absence of further price adjustments prices evolve as $P_{t+s}(i) = \chi_{t,t+s} P_t^*(i)$:

$$\chi_{t,t+s} = \begin{cases} 1 & s = 0, \\ \prod_{l=1}^{s} \pi_{t+l-1}^{\iota_p} \bar{\pi}^{1-\iota_p} & s = 1, \ldots, \infty. \end{cases}$$

Therefore and using the definition $\mathrm{y}_{t+s}(i) = \frac{\mathrm{Y}_{t+s}(i)}{\mathrm{Y}_{t+s}}$:

$$\frac{d(Y_{t+s}(i)[P_{t+s}(i) - MC_{t+s}(i)])}{d\tilde{P}_t(i)} = \mathrm{y}_{t+s}(i) Y_{t+s} \left(\chi_{t,t+s}[1 - \eta_p(\mathrm{y}_{t+s}(i))] + \eta_p \frac{MC_{t+s}(i)}{P_t(i)} \right).$$

The first order condition is then given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s} P_t}{\xi_t P_{t+s}} \mathrm{y}_{t+s}(i) Y_{t+s} \left([1 - \eta_p(\mathrm{y}_{t+s}(i))] \chi_{t,t+s} + \eta_p \frac{MC_{t+s}(i)}{P_t(i)} \right) = 0 \tag{9.13}$$

For future reference, it is useful to re-write the FOC as follows:

$$\frac{P_t^*(i)}{P_t} = \frac{\mathbb{E}_t \sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s \frac{\xi_{t+s}}{\lambda_p(\mathrm{y}_{t,t+s}(i))\xi_t} y_{t,t+s}(i) \frac{\eta_p(\mathrm{y}_{t,t+s}(i))}{\eta_p(\mathrm{y}_{t,t+s}(i))-1} mc_{t+s}(i)}{\mathbb{E}_t \sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s \frac{\xi_{t+s}}{\lambda_p(\mathrm{y}_{t,t+s}(i))\xi_t} \frac{\chi_{t,t+s}}{\prod_{l=1}^{s}\pi_{t+s}} y_{t,t+s}(i)} \tag{9.14}$$

where $\mathrm{y}_{t,t+s}(i) = G'^{-1}\left(\frac{P_t^*\chi_{t,t+s}Y_{t+s}}{\Xi_{t+s}^f}\right), Y_{t,t+s}(i) = \mathrm{y}_{t,t+s}(i)Y_{t+s}$.

Noting that measure $1-\zeta_p$ of firms changes prices in each period and each firm faces a symmetric problem, the expression for the aggregate price index (9.2) can be expressed recursively as a weighted average of adjusted and indexed prices:

$$P_t = (1-\zeta_p)P_t^* G'^{-1}\left(\frac{P_t^* Y_t}{\Xi_t^f}\right) + \zeta_p \pi_{t-1}^{\iota_p}\bar{\pi}^{1-\iota_p} P_{t-1} G'^{-1}\left(\frac{\pi_{t-1}^{\iota_p}\bar{\pi}^{1-\iota_p} P_{t-1} Y_t}{\Xi_t^f}\right), \tag{9.15}$$

using that price distribution of non-adjusting firms at t is the same as that of all firms at time $t-1$, adjusted for the shrinking mass due price adjustments. Along the deterministic balanced growth path the optimal price equals the average price, which is normalized to unity:

$$\bar{P}^* = \bar{P} = 1.$$

Similarly, along the deterministic growth path the price is a constant mark-up over marginal cost:

$$\frac{\bar{P}^*}{\bar{P}} = \frac{\eta_p}{\eta_p - 1}\overline{mc} = (1+\bar{\lambda}_p)\overline{mc} = 1 \tag{9.16}$$

Finally, the assumption of monopolistic competition in the presence of free entry requires zero profits along the balanced growth path. Real and detrended profits of intermediate producer i are given by:

$$\Pi_t^p(i) = \frac{P_t(i)}{P_t} y_t(i) - w_t n_t(i) - r_t^k k_t(i) = \frac{P_t(i)}{P_t} y_t(i) - mc_t(i)[y_t(i) + \mu^t \Phi]$$

Integrating over all $i \in [0,1]$ and using the definition of the price index (9.2) yields:

$$\Pi_t^p = y_t - w_t \int_0^1 n_t(i)di - r_t^k \int_0^1 k_t(i)di \tag{9.17a}$$

$$= y_t - mc_t\bigg(\int_0^1 y_t(i)di + \Phi\bigg) = y_t - mc_t\bigg(y_t \int_0^1 \frac{P_t(i)}{P_t}di + \Phi\bigg) \tag{9.17b}$$

Using the expression for the steady state markup, equation (9.16), the zero profit condition (9.17b) implies that along the symmetric balanced growth path:

$$0 = \bar{\Pi}^p = \bar{y} - \frac{\bar{y}\int_0^1 \frac{P(i)}{P}di + \Phi}{1+\bar{\lambda}_p} = \bar{y} - \frac{\bar{y}+\Phi}{1+\bar{\lambda}_p} \quad \Rightarrow \frac{\Phi}{\bar{y}} = \bar{\lambda}_p. \tag{9.18}$$

9.1.3 Labor packers

Intermediate producers use a bundel of differentiated labor inputs, $\ell \in [0,1]$, purchased from labor packers. Labor packers aggregagte, or pack, differentiated labor which they purchase from unions. They are perfectly competitive and face an analogous problem to final goods producers:

$$\max_{n_t, n_t(\ell)} W_t n_t - \int_0^1 W_t(\ell) n_t(\ell) d\ell \quad \text{s.t.} \quad \int_0^1 H\left(\frac{n_t(\ell)}{n_t}; \tilde{\epsilon}_t^{\lambda,w}\right) d\ell = 1, \tag{9.19}$$

where $H(\cdot)$ has the same properties as $G(\cdot)$: $H' > 0, H'' < 0, H(1) = 1$.

The FOC yield differentiated labor demand, analogous to intermediate goods demand (9.3):

$$n_t(\ell) = n_t H'^{-1}\bigg(\frac{W_t(\ell) n_t}{\Xi_t^n}\bigg) = n_t H'^{-1}\bigg(\frac{W_t(\ell)}{W_t}\int_0^1 H'\bigg(\frac{n_t(l)}{n_t}; \tilde{\epsilon}_t^{\lambda,w}\bigg)\frac{n_t(l)}{n_t}dl\bigg). \tag{9.20}$$

Given the aggregate nominal wage $W_t = \int_0^1 \frac{n_t(\ell)}{n_t} w_t(\ell) d\ell$, labor packers are willing to supply any amount of packed labor n_t. Labor demand elasticity behaves analogously to the intermediate goods elasticity:

$$\eta_w(\mathrm{n}_t(\ell)) \equiv -\frac{W_t(\ell)}{n_t(\ell)}\frac{\mathrm{d}\mathrm{n}_t(\ell)}{dW_t(\ell)}\bigg|_{dn_t = d\Xi_t^l = 0} = -\frac{H'(\mathrm{n}_t(\ell))}{\mathrm{n}_t(\ell) H''(\mathrm{n}_t(\ell))} \tag{9.21}$$

$$\hat{\eta}_w(\mathrm{n}_t(\ell)) \equiv \frac{W_t(\ell)}{\eta_w(\mathrm{n}_t(\ell))}\frac{d\eta_w(\mathrm{n}_t(\ell))}{dW_t(\ell)} = 1 + \frac{1+\lambda^w(\mathrm{n}_t(\ell))}{\lambda^w(\mathrm{n}_t(\ell))}\left(\frac{1}{[1+\lambda^w(\mathrm{n}_t(\ell))]A_w(\mathrm{n}_t(\ell))} - 1\right), \tag{9.22}$$

where $\mathrm{n}_t(\ell) \equiv \frac{n_t(\ell)}{n_t}$ and the mark-up is defined as $\lambda_t^n(\mathrm{n}_t(\ell)) \equiv \frac{1}{\eta_w(\mathrm{n}_t(\ell))-1}$. $A_w(\mathrm{n}_t(\ell)) \equiv \frac{\lambda^w(\mathrm{n}_t(\ell))}{2+\frac{H'''(\mathrm{n}_t(\ell))}{H''(\mathrm{n}_t(\ell))}\mathrm{n}_t(\ell)}$ can be equivalently expressed as:

$$A_w(\mathrm{n}) = \frac{1}{\lambda^w(\mathrm{n})\hat{\eta}_w(\mathrm{n})+1}.. \tag{9.23}$$

9.2 Households

There is a measure one of households in the economy, indexed by $j \in [0,1]$, endowed with a unit of labor each. Households are distributed uniformly over the real line, i.e. the measure of households is the Lebesgue measure Λ. We distinguish two types of households – intertemporally optimizing households $j \in [0, 1-\phi]$ and "rule-of-thumb" households $j \in (1-\phi, 1]$, so that they have measures $\Lambda([0, 1-\phi]) = 1-\phi$ and $\Lambda([0,\phi]) = \phi$, respectively.

Households' preferences over consumption and hours worked streams $\{C_{t+s}(j), n_{t+s}(j)\}_{s=0}^{\infty}$ are represented by the life-time utility function U_t:

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma}\big(C_{t+s}(j) - hC_{t+s-1}\big)^{1-\sigma}\right] \exp\left[\frac{\sigma-1}{1+\nu} n_{t+s}(j)^{1+\nu}\right]. \tag{9.24}$$

Here $h \in [0,1)$ captures external habit formation, σ denotes the inverse of the intertemporal elasticity of substitution, and ν equals the inverse of the labor supply elasticity. Households discount the future by $\beta \in (0,1)$, where β varies by household type.

The fraction $1-\phi$ of the labor force who are not credit constrained, maximizes their life-time utility subject to a lifetime budget constraint and a capital accumulation technology. The remainder of the labor force, i.e. a fraction ϕ is credit constrained (or "rule-of-thumb"): they cannot save or borrow.

9.2.1 Intertemporally optimizing households

The intertemporally optimizing households choose consumption $\{C_{t+s}(j)\}$, investment in physical capital $\{X_{t+s}(j)\}$, physical capital $\{K^p_{t+s}(j)\}$, a capacity utilization rate $\{u_{t+s}(j)\}$, nominal government bond holdings $B^n_{t+s}(j)$, and labor supply $\{n_{t+s}(j)\}$ to maximize (9.24) subject to a sequence of budget constraints (9.25), the law of motion for physical capital (9.26), and a no-Ponzi constraint. Households take prices $\{P_{t+s}\}$, nominal returns on government bonds $\{q^b_{t+s}R_{t+s}\}$, the nominal rental rate of capital $\{R^k_{t+s}\}$, and nominal wages $\{W_{t+s}\}$ as given.

The budget constraint for period $t+s$ is given by:

$$(1+\tau^c_{t+s})C_{t+s}(j) + X_{t+s}(j) + \frac{B^n_{t+s}(j)}{R^{gov}_{t+s}P_{t+s}} \leq$$
$$S_{t+s} + \frac{B^n_{t+s-1}(j)}{P_{t+s}} + (1-\tau^n_{t+s})\frac{[W^h_{t+s}n_{t+s}(j) + \lambda_{w,t+s}n_{t+s}W^h_{t+s}]}{P_{t+s}} +$$
$$+\left[(1-\tau^k_{t+s})\left(\frac{R^k_{t+s}u_{t+s}(j)}{P_{t+s}} - a(u_{t+s}(j))\right) + \delta\tau^k_{t+s}\right][(1-\omega^k_{t+s-1})K^p_{t+s-1}(j) + \omega^k_{t+s-1}K^{p,agg}_{t+s-1}] + \frac{\Pi^p_{t+s}\mu^{t+s}}{P_{t+s}}, \quad (9.2$$

where $(\tau^c_{t+s}, \tau^k_{t+s}, \tau^n_{t+s})$ represent taxes on consumption expenditure, capital income, and labor income, respectively. The wage received by households differs from the one charged to labor packers because of union profits – union profits $\lambda_{w,t+s}n_{t+s}W^h_{t+s}$ are taken as given by households. Households also receive nominal lump-sum transfers $\{S_{t+s}\}$. $a(\cdot)$ represents the strictly increasing and strictly convex cost function of varying capacity utilization, whose first derivative in the case of unit capacity utilization is normalized as $a'(1) = \bar{r}^k$.[5] At unit capacity utilization, there is no additional cost: $a(1) = 0$. $\Pi^p_{t+s}\mu^{t+s}$ are nominal profits which households also take as given.

There is a financial market frictions present in the budget constraint. $\omega^k_{t+s} \neq 0$ represents a wedge between between the returns on private and government bonds and is a pure financial market friction – if $\omega^k_{t+s} > 0$ then

[5] $\bar{r}^k$ represents the real steady state return on capital services.

households obtain less than one dollar for each dollar of after tax capital income they receive, representing agency costs. Agency costs are reimbursed directly to unconstrained households, so that the friction has no effect on aggregate resources. This financial market friction is similar to a shock in Smets and Wouters (2003) who introduce it ad hoc in the investment Euler equation and motivate it as a short-cut to model informational frictions which disappear at the steady state.

Physical capital evolves according to the following law of motion:

$$K^p_{t+s}(j) = (1-\delta)K^p_{t+s-1}(j) + q^x_{t+s}\left[1 - S\left(\frac{X_{t+s}(j)}{X_{t+s-1}(j)}\right)\right] X_{t+s}(j), \quad (9.26)$$

where new investment is subject to adjustment costs described by $S(\dot{})$. These costs satisfy $S(\mu) = S'(\mu) = 0, S'' > 0$. The relative price of investment changes over time, as captured by the exogenous $\{q^x_{t+s}\}$ process. Physical capital depreciates at rate δ.

For future reference, note that the effective capital stock is given by the product of capacity utilization and physical capital stock:

$$K^{eff}_{t+s}(j) = K^p_{t+s-1}(j)u_{t+s}(j). \quad (9.27)$$

To obtain the aggregate capital stock, multiply the above quantity by $(1-\phi)$.

The solution to the household's problem is characterized completely by the law of motion for physical capital (9.26) and the following necessary and sufficient first order conditions. To derive these conditions, denote the Lagrange multipliers on the budget constraint (9.25) and the law of motion (9.26) by $\beta^t(\Xi_t, \Xi^k_t)$ – replacing the household index j by a superscript RA.

$$[C_t] \quad \Xi_t(1+\tau^c_t) = \exp\left(\frac{\sigma-1}{1+\nu}(n^{RA}_t)^{1+\nu}\right)[C^{RA}_t - hC^{RA}_{t-1}]^{-\sigma}$$

$$[n_t] \quad \Xi_t(1-\tau^n_t)\frac{W^h_t}{P_t} = \exp\left(\frac{\sigma-1}{1+\nu}(n^{RA}_t)^{1+\nu}\right)(n^{RA}_t)^{\nu}[C^{RA}_t - hC^{RA}_{t-1}]^{1-\sigma}$$

$$
\begin{aligned}
&[B_t] \quad \Xi_t = \beta q_t^b R_t \mathbb{E}_t \left(\frac{\Xi_{t+1}}{P_{t+1}/P_t} \right) \\
&[K_t^p] \quad \Xi_t^k = \beta \mathbb{E}_t \left(\Xi_{t+1} \left[\tilde{q}_t^k \left((1-\tau_{t+s}^k) \left[\frac{R_{t+1}^k}{P_{t+1}} u_{t+1} - a(u_{t+1}) + \delta \tau_{t+1}^k \right] + (1-\delta) \frac{\Xi_{t+1}^k}{\Xi_{t+1}} \right] \right) \\
&[X_t] \quad \Xi_t = \Xi_t^k q_t^x \left(1 - S\left(\frac{X_t^{RA}}{X_{t-1}^{RA}}\right) - S'\left(\frac{X_t^{RA}}{X_{t-1}^{RA}}\right)\left(\frac{X_t^{RA}}{X_{t-1}^{RA}}\right) \right) + \beta \mathbb{E}_t \left(\frac{\Xi_{t+1}^k}{\Xi_t} q_{t+1}^x S'\left(\frac{X_{t+1}^{RA}}{X_t^{RA}}\right)\left(\frac{X_{t+1}^{RA}}{X_t^{RA}}\right)^2 \right) \\
&[u_t] \quad \frac{R_{t+1}^k}{P_t} = a'(u_{t+1}^{RA}).
\end{aligned}
$$

By setting $a'(1) \equiv \bar{r}^k$ we normalize steady state capacity utilization to unity: $\bar{u} \equiv 1$.

For what follows, it is useful to detrend these first order conditions and the law of motion for capital. To that end, use lower case letters to denote detrended and real variables as exemplified in the following definitions:

$$
k_t^{RA} \equiv \frac{K_t^{RA}}{\mu^t}, w_t \equiv \frac{W_t}{P_t \mu^t}, w_t^h \equiv \frac{W_t^h}{P_t \mu^t}, r_t^k \equiv \frac{R_t^k}{P_t}, \xi_t \equiv \Xi_t \mu^{\sigma t}, Q_t \equiv \frac{\Xi_t^k}{\Xi_t}, \bar{\beta} = \beta \mu^{-\sigma}.
$$

μ denotes the gross trend growth rate of the economy. For future reference, note that government expenditure is normalized differently: $g_t = \frac{G_t}{Y \mu^t}$. Substituting in for the normalized variables yields:

$$
\xi_t (1+\tau_t^c) = \exp\left(\frac{\sigma-1}{1+\nu} (n_t^{RA})^{1+\nu} \right) [c_t^{RA} - (h/\mu) c_{t-1}^{RA}]^{-\sigma} \tag{9.29a}
$$

$$
\xi_t (1-\tau_t^n) w_t^h = \exp\left(\frac{\sigma-1}{1+\nu} (n_t^{RA})^{1+\nu} \right) (n_t^{RA})^{\nu} [c_t^{RA} - (h/\mu) c_{t-1}^{RA}]^{1-\sigma} \tag{9.29b}
$$

$$
\xi_t = \bar{\beta} R_t^{gov} \mathbb{E}_t \left(\frac{\xi_{t+1}}{P_{t+1}/P_t} \right) \tag{9.29c}
$$

$$
Q_t = \bar{\beta} \mathbb{E}_t \left(\frac{\xi_{t+1}}{\xi_t} \left[\tilde{q}_t^k \left((1-\tau_{t+1}^k) [r_{t+1}^k u_{t+1} - a(u_{t+1})] + \delta \tau_{t+1}^k \right) + (1-\delta) Q_{t+1} \right] \right) \tag{9.29d}
$$

$$
\begin{aligned}
1 = {} & Q_t q_t^x \left(1 - S\left(\frac{x_t^{RA}\mu}{x_{t-1}^{RA}}\right) - S'\left(\frac{x_t^{RA}\mu}{x_{t-1}^{RA}}\right)\left(\frac{x_t^{RA}\mu}{x_{t-1}^{RA}}\right) \right) \\
& + \bar{\beta} \mathbb{E}_t \left(\frac{\xi_{t+1}}{\xi_t} Q_{t+1} q_{t+1}^x S'\left(\frac{x_{t+1}^{RA}\mu}{x_t^{RA}}\right)\left(\frac{x_{t+1}^{RA}\mu}{x_t^{RA}}\right)^2 \right)
\end{aligned} \tag{9.29e}
$$

$$
r_{t+1}^k = a'(u_{t+1}^{RA}). \tag{9.29f}
$$

The detrended law of motion for physical capital is given by

$$k_t^{p,RA} = \frac{(1-\delta)}{\mu} k_{t-1}^{p,RA} + q_t^x \left[1 - S\left(\frac{x_t^{RA}}{x_{t-1}^{RA}}\mu\right)\right] x_t^{RA}. \tag{9.30}$$

Combining the FOC for consumption and hours worked, gives the static optimality condition for households:

$$\frac{1-\tau_t^n}{1+\tau_t^c} w_t^h = (n_t^{RA})^\nu [c_t^{RA} - (h/\mu) c_{t-1}^{RA}]. \tag{9.31}$$

Combining (9.29a) for two consecutive periods and using (9.29c) gives the consumption Euler equation:

$$\mathbb{E}_t\left(\frac{\xi_{t+1}}{\xi_t}\right) = \mathbb{E}_t\left(\exp\left(\frac{\sigma-1}{1+\nu}\left(\frac{n_{t+1}^{RA}}{n_t^{RA}}\right)^{1+\nu}\right)\left[\frac{c_{t+1}^{RA} - (h/\mu)c_t^{RA}}{c_t^{RA} - (h/\mu)c_{t-1}^{RA}}\right]^{-\sigma}\right). \tag{9.32}$$

Equation (9.29d) is the investment Euler equation. The FOC for capital (9.29e) can be used to compute the shadow price of physical capital Q_t.

Using the investment Euler equation shows that along the deterministic balanced growth path the value of capital equals unity (since $S'(\mu) = S(\mu) = 0$ and $\bar{q}^x = 1$). From the consumption Euler equation and $\bar{q}^b = 1$ we obtain the interest rate paid on government bonds under balanced growth. Finally, the pricing equation for capital and the investment Euler equation pin down the rental rate on capital. Summarizing:

$$\bar{Q} = 1, \tag{9.33a}$$

$$\bar{R} = \bar{\beta}^{-1}\bar{\pi}, \tag{9.33b}$$

$$1 = \bar{\beta}[(1-\bar{\tau}^k)\bar{r}^k + \delta\bar{\tau}^k + (1-\delta)],$$

$$\Leftrightarrow \quad \bar{r}^k = \frac{\bar{\beta}^{-1} - 1 + \delta(1-\bar{\tau}^k)}{1-\bar{\tau}^k}. \tag{9.33c}$$

The bond premium shock q_t^b differs from a discount factor shock, although it results in an observationally equivalent consumption Euler equation – if time preference was time-varying, the period utility function would become:

$$\left[\frac{1}{1-\sigma}\big(C_{t+s}(j)-hC_{t+s-1}\big)^{1-\sigma}\right]\exp\left[\frac{\sigma-1}{1+\nu}n_{t+s}(j)^{1+\nu}\right]\prod_{l=1}^{s}\check{q}^{b}_{t+l-1},$$

so that the ratio $\frac{\check{\xi}_{t+1}}{\xi_t}$ would be proportional to $\check{q}^b_t$, so that the consumption Euler equation conditions is unchanged. The effects differ, however, insofar that the present formulation on basis of the government discount factor also affects the investment Euler equation and the government budget constraint.

For measurement purposes, it is useful to re-write the linearized FOC for capital, after substituting out for the discount factor. It shows that the private bond shock represents the premium paid for private bonds over government bonds holding the rental rate on capital fixed:

$$\frac{\bar{r}^k(1-\bar{\tau}^k)\mathbb{E}_t(\hat{r}^k_{t+1})+(1-\delta)\mathbb{E}_t(\hat{Q}_{t+1})}{\bar{r}^k(1-\bar{\tau}^k)+\delta\bar{\tau}^k+1-\delta}-\hat{Q}_t=\Big(\hat{R}_t-\mathbb{E}_t[\pi_t]\Big)+\hat{q}^b_t+\hat{q}^k_t.$$

Note: the shock $\tilde{q}^k_t$ in the budget constraint has been rescaled here. $\hat{q}^k_t$ is the deviation of the rescaled shock from its steady state value.

9.2.2 Credit-constrained or "rule of thumb" households

A fraction $\phi \in (0, 0.5)$ of the households is assumed to be credit-constrained. As a justification, one may suppose that credit-constrained discount the future substantially more steeply, and are thus uninterested in accumulating government bonds or private capital, unless their returns are extraordinarily high. Conversely, these households find it easy to default on any loans, and are therefore not able to borrow. We hold the identity of credit-constrained households and thereby their fraction of the total population constant.

"Rule of thumb" households face a static budget constraint in each period and are assumed to supply the same amount of labor as intertemporally optimizing households. Given

$$n^{RoT}_{t+s}(j)=n^{RA}_{t+s}=n_{t+s},$$

consumption follows from the budget constraint in each period:

$$(1+\tau^c_{t+s})C^{RoT}_{t+s}(j) \leq S^{RoT}_{t+s}+(1-\tau^n_{t+s})\frac{W^h_{t+s}n^{RoT}_{t+s}(j)+\lambda_{w,t+s}W^h_{t+s}n_{t+s}}{P_{t+s}}+\Pi^p_{t+s}\mu^{t+s}. \tag{9.34}$$

Rule-of-thumb households receive transfers, labor income including union profits, and profits made by intermediate goods producing firms.

Removing the trend from the budget constraint (9.34), omitting the j index, and solving for (detrended) consumption:

$$c^{RoT}_{t+s} = \frac{1}{(1+\tau^c_{t+s})}\left(s^{RoT}_{t+s}+(1-\tau^n_{t+s})[w^h_{t+s}n^{RoT}_{t+s}+\lambda_{w,t+s}w^h_{t+s}n_{t+s}]+\Pi^p_{t+s}\right). \tag{9.35}$$

From the budget constraint (9.34), the following steady state relationship holds:

$$\bar{c}^{RoT} = \frac{\bar{s}^{RoT}+(1-\bar{\tau}^n_t)\bar{w}\bar{n}}{1+\bar{\tau}^c}. \tag{9.36}$$

We assume that:

$$\bar{s}^{RoT} = \bar{s}. \tag{9.37}$$

9.2.3 Households: labor supply, wage setting

Households supply homogeneous labor to unions which differentiate labor into varieties indexed by $\ell \in [0,1]$ and sell it to labor packers. In doing so, unions take aggregate quantities, i.e. households' cost of supplying labor and aggregate labor demand and wages, as given. Unions maximize the expected present discounted value of net of tax wage income earned in excess of the cost of supplying labor. In the presence of rule-of-thumb households unions act as if they were maximizing surplus for the intertemporally optimizing households only. If the mass of rule-of-thumb households is less than the mass of intertemporally optimizing households, i.e. $\phi < 0.5$ which is satisfied in the parameterizations used, a median-voter decision rule justifies this assumption.

The labor unions problem is analogous to that of price-setting firms, with the marginal rate of substitution between consumption and leisure in the representative household taking the role of marginal costs in firms' problems. From the FOC $[C_t]$ and $[n_t]$ the marginal rate of substitution is given by $\frac{U_{n,t+s}}{\Xi_{t+s}} = (n_t^{RA})^\nu [C_t^{RA} - hC_{t-1}^{RA}](1+\tau_t^c)$. Whenever a union has the chance to reset the wage it charges, it chooses $W_t^*(\ell)$:

$$W_t^*(\ell) = \arg\max_{\tilde{W}_t(\ell)} \mathbb{E}_t \sum_{s=0}^{\infty} (\zeta_w)^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t} \left[(1-\tau_{t+s}^n) \frac{W_{t+s}(\ell)}{P_{t+s}} + \frac{U_{n,t+s}}{\Xi_{t+s}} \right] n_{t+s}(\ell), \tag{9.38}$$

subject to the labor demand equation (9.20). $1-\zeta_w$ denotes the probability that a union can reset its wage. If it cannot adjust, wages are adjusted according to a moving average of past and steady state inflation and labor productivity growth:

$$W_{t+s}(\ell) = {W^*}_t(\ell) \prod_{v=1}^{s} \mu(\pi_{t+v-1})^{\iota_w} \bar{\pi}^{1-\iota_w} \equiv W_t^*(\ell) \chi_{t,t+s}^w.$$

Using that $n_t = n_t^{RA}$, the first order condition is given by

$$\begin{aligned} 0 = \mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t \lambda^w(\mathrm{n}_{t,t+s}(\ell))} \frac{n_{t+s}(\ell)}{W_t^*(\ell)} \Bigg(& (1-\tau_{t+s}^n) \frac{W_t^*(\ell)\chi_{t,t+s}^w(\ell)}{P_{t+s}} \\ & - [1+\lambda^w(\mathrm{n}_{t+s}(\ell))](1+\tau_{t+s}^c) n_{t+s}^\nu [C_{t+s}^{RA} - hC_{t+s-1}^{RA}] \Bigg) \end{aligned} \tag{9.39}$$

and can be equivalently expressed as

$$\frac{W_t^*(\ell)}{P_t} = \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t \lambda^w(\mathrm{n}_{t,t+s}(\ell))} n_{t+s}(\ell) [1+\lambda^w(\mathrm{n}_{t+s}(\ell))](1+\tau_{t+s}^c) n_{t+s}^\nu [C_{t+s}^{RA} - hC_{t+s-1}^{RA}]}{\mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t \lambda^w(\mathrm{n}_{t,t+s}(\ell))} n_{t+s}(\ell)(1-\tau_{t+s}^n) \frac{\chi_{t,t+s}^w(\ell)}{P_{t+s}/P_t}} \tag{9.40}$$

Aggregate wages evolve as

$$W_t = (1-\zeta_w)W_t^* H'^{-1}\left(\frac{W_t^* n_t}{\Xi_t^n}\right) + \zeta_w \pi_{t-1}^{\iota_w} \bar{\pi}^{1-\iota_w} W_{t-1} H'^{-1}\left(\frac{\pi_{t-1}^{\iota_w}\bar{\pi}^{1-\iota_w} W_{t-1} n_t}{\Xi_t^n}\right), \tag{9.41}$$

Along the deterministic balanced growth path, the detrended desired real wage is given by a constant mark-up over the marginal rate of substitution. Given constant inflation, the symmetric deterministic growth path also implies, from equation (9.41), that the desired real wage equals the actual real wage:

$$\bar{w} = \bar{w}^* = (1+\bar{\lambda}_w)\bar{w}^h = (1+\bar{\lambda}_w)\frac{1+\bar{\tau}^c}{1-\bar{\tau}^n}\bar{n}^\nu \bar{c}^{RA}[1-h/\mu], \tag{9.42}$$

where the second equality uses (9.31).

9.3 Government

The government sets nominal interest R_t according to an interest rate rule, purchases goods and services for government consumption G_t, pays transfers S_t to households, and provides public capital for the production of intermediate goods, K_t^g. It finances its expenditures by levying taxes on capital and labor income, a tax on consumption expenditure, and one period nominal bond issues. We consider a setup in which monetary policy is active in the neighborhood of the balanced growth path.

9.3.1 Fiscal policy

In modelling the government sector, we take as given the tax structure along the balanced growth path as in **?**, who used NIPA data to compute the capital and labor income and comsumption expenditure tax rates for the US. Off the balanced growth path, we follow **?** in assuming that labor tax rates adjust gradually to balance the budget in the long-run, whereas in the short-run much of any additional government expenditure is tax financed.

The government flow budget constraint is given by:

$$G_t + X_t^g + S_t + \frac{B_{t-1}}{P_t} \leq \frac{B_t}{R_t^{gov} P_t} + \tau_t^c C_t + \tau_t^n n_t \frac{W_t}{P_t} + \tau_t^k \left[u_t \frac{R_t^k}{P_t} - a(u_t) - \delta \right] K_{t-1}^p. \tag{9.43}$$

Detrended, the government budget constraint is given by:

$$\bar{y} g_t + x_t^g + s_t + \frac{b_{t-1}}{\mu \pi_t} \leq \frac{b_t}{R_t^{gov}} + \tau_t^c c_t + \tau_t^n n_t w_t + \tau_t^k k_t^s r_t^k - \tau_t^k [a(u_t) + \delta] \frac{k_{t-1}^p}{\mu}. \tag{9.44}$$

Government consumption $g_t = \frac{G_t}{\bar{y} \mu^t}$ is given exogenously and is stochastic, driven by genuine spending shocks as well as by technology shocks.

By introducing a wedge between the federal funds rate and government bonds, we capture both short-term liquidity premia as well as changes in the term structure of government debt. Since the latter is absent with only one period bonds, in the estimation the bond premium may also reflect differences in the borrowing cost due to a more complex maturity structure. [6]

Labor tax rates have both a stochastic and a deterministic component. They adjust deterministically to ensure long-run budget balance at a speed governed by the parameter $\psi_\tau \in [\underline{\psi}_\tau, 1]$, where $\underline{\psi}_\tau$ is some positive number large enough to guarantee stability. To simplify notation denote the remaining detrended deficit prior to new debt and changes in labor tax rates as d_t:

$$d_t \equiv \bar{y} g_t + x_t^g + \bar{s} + s_t^{exo} + \frac{b_{t-1}}{\mu \pi_t} - \bar{\tau}^c c_t - \bar{\tau}^n w_t n_t - \bar{\tau}^k k_t^s r_t^k + \bar{\tau}^k \delta \frac{k_{t-1}^p}{\mu}.$$

In the baseline case, labor tax rates are adjusted according to the following rule:

$$(\tau_t^n - \bar{\tau}^n) w_t n_t + \epsilon_t^\tau = \psi_\tau (d_t - \bar{d}), \tag{9.45}$$

[6]Historical data by the Federal Reserve implies a maturity between 10 and 22 quarters with an average between 16 and 20 quarters (The Federal Reserve Board Bulletin, 1999, Figure 4).

where ϵ_t^τ is an exogenous shock to the tax rate.

In general:

$$\psi_\tau(d_t - \bar{d}) - \epsilon_t^\tau = \begin{cases} (\tau_t^n - \bar{\tau}^n) w_t n_t & \text{Baseline}, \tau_t^c = \tau_t^k = s_t^{endo} = 0, \\ (\tau_t^c - \bar{\tau}^c) c_t & \text{Alternative 1}, \tau_t^n = \tau_t^k = s_t^{endo} = 0, \\ (\tau_t^k - \bar{\tau}^k) k_t^s (r_t^k - \delta) & \text{Alternative 2}, \tau_t^n = \tau_t^c = s_t^{endo} = 0, \\ -(s_t^{endo} - \bar{s}) & \text{Alternative 3}, \tau_t^n = \tau_t^c = \tau_t^k = 0. \end{cases} \tag{9.46}$$

Debt issues are then given by the budget constraint or equivalently as the residual from (9.45): $\frac{b_t}{R_t^{gov}} = (1 - \psi_\tau)(d_t - \bar{d}) + \epsilon_t^\tau$.

Government investment is chosen optimally for a given tax structure. Given the congestion effect of production on public infrastructure, a tax on production would be optimal (Barro and Sala-i Martin, 1992). Similarly, we neglect the potential cost of financing of productive government expenditure via distortionary taxes. To motivate this assumption note that along the balanced growth path, government capital can be completely debt-financed or privatized and financed through government bond issues, whereas other government expenditures such as transfers which are not backed by real assets have to backed by the government's power to levy taxes.

Formally, the government chooses investment and capital stock to maximize the present discounted value of output net of investment expenditure along the balanced growth path:

$$\max_{\{K_{t+s}^g, X_{t+s}^g\}_{s=0}^\infty} \mathbb{E}_t \sum_{s=0}^\infty \beta^s \frac{\Xi_{t+s}}{\Xi_t} [Y_{t+s} - X_{t+s}^g],$$

given K_{t-1}^g and subject to the aggregate production function (9.8) and to the capital accumulation equation

$$K_{t+s}^g = (1-\delta) K_{t+s-1}^g + q_{t+s}^{x,g} \left[1 - S_g\left(\frac{[X_{t+s}^g + \tilde{u}_{t+s}^{x,g}]}{[X_{t+s-1}^g + \tilde{u}_{t+s-1}^{x,g}]} \right) \right] (X_{t+s}^g + \tilde{u}_{t+s}^{x,g}). \tag{9.47}$$

The government is subject to similar adjustment costs as the private sector

$S_g(\mu) = S_g'(\mu) = 0, S_g'' > 0$ and investment is subject to shocks to its relative efficiency $q_{t+s}^{x,g}$. We assume that government capital depreciates at the same rate as private physical capital. $\tilde{u}^{x,g}$ represents exogenous shocks to government investment spending – such as stimulus spending.

Denote the Lagrange multiplier on (9.47) at time $t+s$ as $\beta^s \frac{\Xi_{t+s}^g}{\Xi_t}$. Then the first order conditions are:

$$
\begin{aligned}
[X_t^g] \quad 1 = {} & \frac{\Xi_t^g}{\Xi_t} q_t^x \left(1 - S_g\Big(\frac{[\tilde{u}_t^{x,g} + X_t^g]}{[\tilde{\epsilon}_{t-1}^{x,g} + X_{t-1}^g]}\Big) - S_g'\Big(\frac{[\tilde{\epsilon}_t^{x,g} + X_t^g]}{[\tilde{\epsilon}_{t-1}^{x} + X_{t-1}^g]}\Big)\Big(\frac{[\tilde{u}_t^{x,g} + X_t^g]}{[\tilde{\epsilon}_{t-1}^{x,g} + X_{t-1}^g]}\Big)\right) \\
& + \beta \mathbb{E}_t \left(\frac{\Xi_{t+1}^g}{\Xi_t} q_{t+1}^x S_g'\Big(\frac{[\tilde{\epsilon}_{t+1}^{x,g} + X_{t+1}^g]}{[\tilde{u}_t^{x,g} + X_t^g]}\Big)\Big(\frac{[\tilde{\epsilon}_{t+1}^{x,g} + X_{t+1}^g]}{[\tilde{u}_t^{x,g} + X_t^g]}\Big)^2\right) \\
[K_t^g] \quad \frac{\Xi_t^g}{\Xi_t} = {} & \beta \mathbb{E}_t \left(\frac{\Xi_{t+1}}{\Xi_t} \zeta \frac{Y_t + \mu^t \Phi}{K_{t-1}^g} + (1-\delta) \frac{\Xi_{t+1}^g}{\Xi_t}\right)
\end{aligned}
$$

Defining the shadow price of government capital as $Q_t^g \equiv \frac{\Xi_t^g}{\Xi_t}$ and detrending, the first order conditions can be equivalently written as:

$$
\begin{aligned}
1 = {} & Q_t^g q_t^x \left(1 - S_g\Big(\frac{[\epsilon_t^{x,g} +{}_t^g]\mu}{[\epsilon_{t-1}^{x,g} + x_{t-1}^g]}\Big) - S_g'\Big(\frac{[\epsilon_t^{x,g} + x_t^g]\mu}{[\tilde{\epsilon}_{t-1}^{x} + x_{t-1}^g]}\Big)\Big(\frac{[\epsilon_t^{x,g} + x_t^g]\mu}{[\epsilon_{t-1}^{x,g} + x_{t-1}^g]}\Big)\right) \\
& + \bar{\beta} \mathbb{E}_t \left(Q_{t+1}^g \frac{\xi_{t+1}}{\xi_t} q_{t+1}^x S_g'\Big(\frac{[\epsilon_{t+1}^{x,g} + x_{t+1}^g]\mu}{[\epsilon_t^{x,g} + x_t^g]}\Big)\Big(\frac{[\tilde{\epsilon}_{t+1}^{x,g} + x_{t+1}^g]\mu}{[\epsilon_t^{x,g} + x_t^g]}\Big)^2\right)
\end{aligned} \tag{9.48a}
$$

$$
Q_t^g = \bar{\beta} \mathbb{E}_t \left(\frac{\xi_{t+1}}{\xi_t} \zeta \frac{y_t + \Phi}{k_{t-1}^g/\mu} + \frac{\xi_{t+1}}{\xi_t} (1-\delta) Q_{t+1}^g\right), \tag{9.48b}
$$

where $\epsilon_t^{x,g} \equiv \frac{1}{\mu} \tilde{\epsilon}_t^{x,g}$ denotes the detrended investment spending shock.

Along the balanced growth path, $S_g(\mu) = S_g'(\mu) = 0, \bar{q}^{x,g} = 1, \bar{\epsilon}^{x,g} = 0$ ensure that the shadow price of capital equals unity. Introduce r_t^g as a shorthand for the implied rental rate on government capital:

$$
r_t^g = \zeta \frac{y_t + \Phi}{k_t^g/\mu}. \tag{9.49}
$$

In the steady state, from (9.48b):

$$
\bar{r}^g = \bar{\beta}^{-1} - (1-\delta) \tag{9.50}
$$

Equation (9.48b) determines the optimal ratio of government capital to gross output. Importantly, the law of motion for government capital (9.47) and (9.48b) evaluated at the balanced growth path allow to back out the share of government capital in the aggregate production function, for any given government investment to net output ratio $\frac{\bar{x}^g}{\bar{y}}$. From the law of motion along the balanced growth path:

$$\bar{x}^g = \left(1 - \frac{1-\delta}{\mu}\right)\bar{k}^g \quad \Leftrightarrow \quad \frac{\bar{x}^g}{\bar{y}} = [\mu - (1-\delta)frac\bar{k}^g\mu\bar{y}$$

From the equation for r_t^g we have that $\frac{\bar{k}^g}{\mu\bar{y}} = \zeta\frac{\bar{y}+\Phi}{\bar{y}}\frac{1}{\bar{r}^g}$. Combined with the previous equation this allows to solve for the government capital share ζ:

$$\zeta = \frac{\bar{y}}{\bar{y}+\Phi}\frac{\bar{r}^g}{1-(1-\delta)}\frac{\bar{x}}{\bar{y}} \tag{9.51}$$

9.3.2 Monetary policy

The specification of the interest rate rule follows Smets and Wouters (2007). The Federal Reserve sets interest rates according to the following rule:

$$\frac{R_t^{FFR}}{\bar{R}} = \left(\frac{R_{t-1}^{FFR}}{\bar{R}}\right)^{\rho_R}\left[\left(\frac{\pi_t}{\bar{\pi}}\right)^{\psi_1}\left(\frac{Y_t}{Y_t^f}\right)^{\psi_2}\right]^{1-\rho_R}\left(\frac{Y_t/Y_{t-1}}{Y_t^f/Y_{t-1}^f}\right)^{\psi_3}\epsilon_t^r, \tag{9.52}$$

where ρ_R determines the degree of interest rate smoothing and Y_t^f denotes the level of output that would prevail in the economy in the absence of nominal frictions and with constant markups, i.e. the flexible output level. $\psi_1 > 1$ determines the reaction to inflation to deviations of inflation from its long-run average and $\psi_2, \psi_3 > 0$ determine the reaction to the deviation of actual output from the flexible economy output and to the change in the gap between actual and flexible output.

Due to financial market frictions, the return on government bonds differs from the federal funds rate:

$$R_t^{gov} = R_t^{FFR}(1+\omega_t^b)$$

The flexible economy is the limit point of the economy characterized above with $\zeta_p = \zeta_w = 0$ and no markup shocks: $\epsilon_t^{\lambda,p} = \epsilon_t^{\lambda,w} = 0$. From the pricing and wages setting rules this limiting solution implies:

$$\frac{P_t^f(i)}{P_t^f} = [1 + \lambda_p(\mathrm{y}_t^f(i))]mc_t^f(i), \tag{9.53}$$

$$\frac{W_t^f(\ell)}{P_t^f} = [1 + \lambda_w(\mathrm{n}_t^f(\ell))]\frac{1+\tau_t^c}{1-\tau_t^{n,f}}{n_t^f}^{\nu}[C_t^f - hC_{t-1}^f], \tag{9.54}$$

where the superscript f denotes variables in the flexible economy. Given that final goods are the numeraire and given that firms are symmetric and can freely set their prices:

$$1 = P_t^f = P_t^f(i) = [1 + \lambda_p(1)]mc_t^f(i) \quad \forall t, \tag{9.55}$$

implying that marginal costs are constant for all firms.

Similarly, since all unions face a symmetric problem and can freely reset wages we have that, using that the numeraire equals unity and diving be trend growth:

$$\frac{W_t^f(\ell)}{\mu} = \frac{W_t^f}{\mu} = w_t^f = [1 + \lambda_w(1)]\frac{1+\tau_t^c}{1-\tau_t^{n,f}}{n_t^f}^{\nu}[c_t^f - (h/\mu)c_{t-1}^f]. \tag{9.56}$$

Money does not enter explicitly in the economy: the Federal Reserve supplies the amount of money demanded at interest rate R_t.

9.4 Exogenous processes

The exogenous processes are assumed to be log-normally distributed and, with the exception of government spending shocks, to be independent. Government spending shocks are correlated with technology shocks. Shocks to the two mark-up processes follow an ARMA(1,1) process, whereas the other shocks are AR(1) processes.

$$\log \epsilon_t^a = \rho_a \log \epsilon_{t-1}^a + u_t^a, \qquad u_t^a \overset{iid}{\sim} \mathcal{N}(0, \sigma_a^2) \tag{9.57a}$$

$$\log \epsilon_t^r = \rho_r \log \epsilon_{t-1}^r + u_t^r, \qquad u_t^r \overset{iid}{\sim} \mathcal{N}(0, \sigma_r^2) \tag{9.57b}$$

$$\log g_t = \log g_t^a + \tilde{u}_t^g, \tag{9.57c}$$

$$\log g_t^a = (1 - \rho_g) \log \bar{g} + \rho_g \log g_{t-1}^a + \sigma_{ga} u_t^a + u_t^g, \qquad u_t^a \overset{iid}{\sim} \mathcal{N}(0, \sigma_a^2) \tag{9.57d}$$

$$\log s_t^{exo} = \tilde{u}_t^s, \tag{9.57e}$$

$$\log \epsilon_t^\tau = \rho_\tau \log \epsilon_{t-1}^\tau + u_t^\tau, \qquad u_t^\tau \overset{iid}{\sim} \mathcal{N}(0, \sigma_\tau^2) \tag{9.57f}$$

$$\log \tilde{\epsilon}_t^{\lambda,p} = \rho_{\lambda,p} \log \tilde{\epsilon}_{t-1}^{\lambda,p} + u_t^{\lambda,p} - \theta_{\lambda,p} u_{t-1}^{\lambda,p}, \qquad u_t^{\lambda,p} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\lambda,p}^2) \tag{9.57g}$$

$$\log \tilde{\epsilon}_t^{\lambda,w} = \rho_{\lambda,w} \log \tilde{\epsilon}_{t-1}^{\lambda,w} + u_t^{\lambda,w} - \theta_{\lambda,w} u_{t-1}^{\lambda,w}, \qquad u_t^{\lambda,w} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\lambda,w}^2) \tag{9.57h}$$

$$\log(1 + \omega_t^b) \equiv \log q_t^b = \rho_b \log q_{t-1}^b + u_t^b, \qquad u_t^b \overset{iid}{\sim} \mathcal{N}(0, \sigma_b^2) \tag{9.57i}$$

$$\log(1 - \omega_t^k) \equiv \log q_t^k = \rho_k \log q_{t-1}^k + u_t^k, \qquad u_t^k \overset{iid}{\sim} \mathcal{N}(0, \sigma_k^2) \tag{9.57j}$$

$$\log q_t^x = \rho_x \log q_{t-1}^x + u_t^x, \qquad u_t^x \overset{iid}{\sim} \mathcal{N}(0, \sigma_x^2) \tag{9.57k}$$

$$\log q_t^{x,g} = \rho_{x,g} \log q_{t-1}^{x,g} + u_t^{x,g}, \qquad u_t^{x,g} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{x,g}^2) \tag{9.57l}$$

Three shocks are deterministic and used for policy counterfactuals only:

$$\tilde{u}_t^s, \tilde{u}_t^g, \tilde{u}_t^{x,g}.$$

9.5 Equilibrium conditions

9.5.1 Aggregation

From the final goods producers' problem (9.1) and using the zero profit condition in the competitive market, net output in nominal and real terms is given by

$$P_t Y_t = \int_0^1 P_t(i) Y_t(i) di \quad \Leftrightarrow \quad Y_t = \int_0^1 \frac{P_t(i)}{P_t} Y_t(i) di.$$

Outside the flexible economy, relative prices differ from unity, so that output is not simply the average production of intermediates. However, to a first order price dispersion is irrelevant because $y_t(i) \approx y_t - \eta_p(1) y_t \left(\frac{P_t(i)}{P_t} - 1 \right)$, so that the dispersion term averages out in the aggregate $\int_0^1 y_t(i) di \approx y_t$.

In the presence of heterogeneous labor, the measurement of labor supply faces similar issues because

$$n_t = \int_0^1 \frac{W_t(\ell)}{W_t} n_t(\ell) d\ell,$$

which, by analogy to the above argument for output, generally differs from average hours. However, to a first order:

$$\int_0^1 n_t(\ell) d\ell \approx n_t \tag{9.58}$$

Non-credit constrained households are indexed by $j \in [0, 1-\phi]$ and there is measure $1-\phi$ of these households in the economy. Each non-credit constraint household supplies $K_t(j) = K_t^{RA}$ units of capital services, so that total holdings of capital capital and government bonds per intertemporally optimizing household are given by $\frac{1}{1-\phi}$ times the aggregate quantity. Similarly, household investment is a multiple of aggregate investment. To see this, note that aggregate quantities of bond holdings B_t, investment X_t, physical capital K_t^p, and capital services K_t are computed as:

$$K_t = \int_0^{1-\phi} K_t(j)\Lambda(dj) = K_t(1-\phi)^{-1}\Lambda([0,1-\phi]) = K_t.$$

Aggregate consumption is given by:

$$C_t = \int_0^1 C_t(j)\Lambda(dj) = \int_0^{1-\phi} C_t^{RA}\Lambda(dj) + \int_{1-\phi}^1 C_t^{RoT}\Lambda(dj) = (1-\phi)C^R A_t + \phi C_t^{RoT}. \tag{9.59}$$

Given the consumption of rule-of-thumb agents (9.36), that of intertemporally optimizing agents is given by:

$$\bar{c}^{RA} = \frac{\bar{c} - \phi\bar{c}^{RoT}}{1-\phi}. \tag{9.60}$$

Similarly, aggregate transfers are given by

$$S_t = (1-\phi)S_t^{RA} + \phi S_t^{RoT}, \tag{9.61}$$

where equation (9.37) implies that:

$$\bar{s} = \bar{s}^{RA} + \bar{s}^{RoT}.$$

Aggregate labor supply coincides with individual labor supply of either type of household.

9.5.2 Market Clearing

Labor market clearing requires that labor demanded by intermediaries equals labor supplied by labor packers:

$$\int_0^1 n_t(i)di = n_t = n_t \int_0^1 \frac{W_t(\ell)}{W_t} n_t(\ell)d\ell,$$

where $n_t(\ell)$ is measured in units of the differentiated labor supplies and n_t is measured in units which differs from those supplied by households.

Adding the government and the budget constraints of the two types of households, integrated over $[0, 1-\phi]$ and $(1-\phi, 1]$, respectively, and substituting $\int_0^1 n_t(j) W_t^h (1+\lambda_{t,w}) dj = W_t n_t$, which results from combining the labor packers' zero profit condition with the union problem into the household budget constraint, yields the following equation:

$$C_{t+s} + X_{t+s}(j) + G_t + X_{t+s}^g = n_t \frac{W_{t+s}}{P_{t+s}} + \left[\frac{R_{t+s}^k u_{t+s}}{P_{t+s}} - a(u_{t+s}) \right] K_{t+s-1}^p + \frac{\Pi_{t+s}^p \mu^{t+s}}{P_{t+s}},$$

Detrending and substituting in for real profits from (9.17a), using that $w_t \int_0^1 n_t(i) di = w_t n_t$:

$$c_{t+s} + x_{t+s} + \bar{y} g_{t+s} + x_{t+s}^g = y_{t+s} - a(u_{t+s}) \mu k_{t+s-1}^p, \tag{9.62}$$

which is the goods market clearing condition: Production is used for government and private consumption, government and private investment, as well as variations in capacity utilization.

9.6 Linearized equilibrium conditions

9.6.1 Firms

Log-linearizing the production function around the symmetric balanced growth path:

$$\hat{y}_t = \frac{\bar{y} + \Phi}{\bar{y}} \left(\hat{\epsilon}_t^a + \zeta \hat{k}_{t-1}^g + \alpha(1-\zeta) \hat{k}_t + (1-\alpha)(1-\zeta) \hat{n}_t \right). \tag{9.63}$$

The capital-labor ratio is approximated by (9.9):

$$\hat{k}_t = \hat{n}_t + \hat{w}_t - \hat{r}_t^k, \tag{9.64}$$

where symmetry around the balanced growth path was used.

Marginal costs in (9.65) are approximated by

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}^k_t - \frac{1}{1-\zeta}\left(\zeta\hat{k}^g_t - \zeta\frac{\bar{y}}{\bar{y}+\Phi}\hat{y}_t + \hat{\epsilon}^a_t\right)\left(\frac{k^g_t}{y_t+\Phi}\right)^{\frac{\zeta}{1-\zeta}}\tilde{\epsilon}^a_t, \tag{9.65}$$

and in the flexible economy from (9.55):

$$\widehat{mc}^f_t = 0 \tag{9.66}$$

To-log linearize the pricing FOC (9.14), note that to a first order the common terms in numerator and denominator, i.e. $\frac{\xi_{t+s} y_{t,t+s}(i)}{\lambda_p(\mathrm{y_{t+s}(i)})\xi_t}$, cancel out, using equation (9.16). As a preliminary step notice that in the absence of mark-up shocks:

$$\begin{aligned}
\overline{mc}d\left(\frac{\eta_p(\mathrm{y}_{t+s}(i))}{1-\eta_p(\mathrm{y}_{t+s}(i))}\right)\Big|_{\mathrm{y}_{t+s}(i)=1} &= \overline{mc}\frac{\bar{\eta}_p}{1-\bar{\eta}_p}\frac{-1}{1-\bar{\eta}_p}\frac{d\eta_p(\mathrm{y}_{t+s}(i))|_{\mathrm{y}_{t+s}(i)=1}}{\bar{\eta}_p} \\
&= -\bar{\lambda}_p\hat{\eta}_p(1)d\left(\frac{P^*_t(i)}{P_{t+s}}\right)\Big|_{\frac{P_{t+s}(i)}{P_{t+s}}=1}, \\
d\left(\frac{P_{t+s}(i)}{P_{t+s}}\right)\Big|_{\frac{P^*_t(i)}{P_{t+s}}=1} &= d\left(\frac{\chi_{t,t+s}}{\prod_{l=1}^s \pi_{t+l}}\right) + d\left(\frac{P^*_t(i)}{P_t}\right).
\end{aligned}$$

Notice that from (9.22):

$$1+\bar{\lambda}_p\hat{\eta}_p = \frac{1}{\bar{A}_p}$$

To simplify notation and to address mark-up shocks use $\bar{\epsilon}^{\lambda,p} = 1$ define

$$\begin{aligned}
p^*_t(i) &\equiv \frac{P^*_t(i)}{P_t}, \\
\hat{\epsilon}^{\lambda,p}_{t+s} &\equiv \frac{\partial}{\partial\epsilon^{\lambda,p}_{t+s}}\left(\frac{\eta_p(\mathrm{y}_{t+s}(i))}{1-\eta_p(\mathrm{y}_{t+s}(i))}\right)\Big|_{\mathrm{y}_{t+s}(i)=1}\hat{\hat{\epsilon}}^{\lambda,p}_{t+s} = \frac{\eta_p(1)}{[1-\eta_p(1)]^2}\left(\frac{G'_\epsilon(1)}{G'(1)} - \frac{G''_\epsilon(1)}{G''(1)}\right).
\end{aligned}$$

Now, taking a first-order approximation of (9.14) and using symmetry yields

$$0 = \mathbb{E}_t\sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s\Big[\hat{p}^*_t(i) + \sum_{l=1}^{s}[\iota_p\hat{\pi}_{t+l-1} - \hat{\pi}_{t+l}]\Big](1+\bar{\lambda}_p\hat{\eta}(1)) - [\widehat{mc}_{t+s} + \hat{\epsilon}^{\lambda,p}_{t+s}]$$

$$
\begin{aligned}
\Leftrightarrow \frac{1}{1-\bar{\beta}\zeta_p\mu}\frac{1}{\bar{A}_p}\hat{p}_t^* &= \mathbb{E}_t\sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s[\widehat{mc}_{t+s}+\hat{\epsilon}_{t+s}^{\lambda,p}] - \sum_{l=1}^{s}[\iota_p\hat{\pi}_{t+l-1}-\hat{\pi}_{t+l}]\frac{1}{\bar{A}_p} \\
&= \widehat{mc}_t + \hat{\epsilon}_t^{\lambda,p} - \frac{\bar{\beta}\mu\zeta_p}{1-\bar{\beta}\mu\zeta_p}\frac{1}{\bar{A}_p}[\iota_p\hat{\pi}_t - \mathbb{E}_t\hat{\pi}_{t+1}] \\
&+ \mu\bar{\beta}\zeta_p\mathbb{E}_t\mathbb{E}_{t+s}\sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s[\widehat{mc}_{t+1+s}+\hat{\epsilon}_{t+1+s}^{\lambda,p}] - \sum_{l=1}^{s}[\iota_p\hat{\pi}_{t+l}-\hat{\pi}_{t+1+l}]\frac{1}{\bar{A}_p} \\
&= \widehat{mc}_t + \hat{\epsilon}_t^{\lambda,p} - \frac{\bar{\beta}\mu\zeta_p}{1-\bar{\beta}\mu\zeta_p}\frac{1}{\bar{A}_p}[\iota_p\hat{\pi}_t - \mathbb{E}_t\hat{\pi}_{t+1}] + \mu\bar{\beta}\zeta_p\mathbb{E}_t\hat{p}_{t+1}^*.
\end{aligned}
$$

Now, linearizing the evolution of the price index (9.15):

$$
\hat{p}_t^* = \frac{\zeta_p}{1-\zeta_p}[\hat{\pi}_t - \iota_p\hat{\pi}_{t-1}] \quad \Leftrightarrow \quad \hat{\pi}_t = \frac{1-\zeta_p}{\zeta_p}\hat{p}_t^* + \iota_p\hat{\pi}_{t-1}.
$$

Forwarding the equation once and substituting in and solving for $\hat{\pi}_t$ yields:

$$
\hat{\pi}_t = \frac{\iota_p}{1+\iota_p\bar{\beta}\mu}\hat{\pi}_{t-1} + \frac{1-\zeta_p\bar{\beta}\mu}{1+\iota_p\bar{\beta}\mu}\frac{1-\zeta_p}{\zeta_p}\bar{A}_p(\widehat{mc}_t+\hat{\epsilon}_t^{\lambda,p}) + \frac{\bar{\beta}\mu}{1+\iota_p\bar{\beta}\mu}\mathbb{E}_t\hat{\pi}_{t+1} \tag{9.67}
$$

9.6.2 Households

The law of motion for capital (9.26) and the fact that individual capital holdings are proportional to aggregate capital holdings implies:

$$
\hat{k}_t^p = \left(1-\frac{\bar{x}}{\bar{k}^p}\right)\hat{k}_{t-1}^p + \frac{\bar{x}}{\bar{k}^p}(\hat{x}_t + \hat{q}_{t+s}^x). \tag{9.68}
$$

From (9.27), capital services evolve as:

$$
\hat{k}_t = \hat{u}_t + \hat{k}_{t-1}^p \tag{9.69}
$$

From the static optimality condition (9.31)

$$
\hat{w}_t^h = \nu\hat{n}_t + \frac{\hat{c}_t^{RA} - (h/\mu)\hat{c}_{t-1}^{RA}}{1-h/\mu} + \frac{d\tau_t^n}{1-\bar{\tau}^n} + \frac{d\tau_t^c}{1+\tau^c}. \tag{9.70}
$$

In the flexible economy, given the absence of mark-up shocks equation (9.56) implies:

$$\hat{w}_t^f = \nu \hat{n}_t^f + \frac{\hat{c}_t^{RA,f} - (h/\mu)\hat{c}_{t-1}^{RA,f}}{1 - h/\mu} + \frac{d\tau_t^{n,f}}{1 - \bar{\tau}^n} + \frac{d\tau_t^{c,f}}{1 + \bar{\tau}^c}. \tag{9.71}$$

In the presence of rigidities, the dynamic wage setting equation (9.40) can be linearized as in the derivation of (9.67), recognizing that the analogue to marginal costs is given by (9.70):[7]

$$\begin{aligned}
\hat{w}_t &= \frac{\hat{w}_{t-1}}{1 + \bar{\beta}\mu} + \frac{\bar{\beta}\mu \mathbb{E}_t[\hat{w}_{t+1}]}{1 + \bar{\beta}\mu} \\
&+ \frac{(1 - \zeta_w \bar{\beta}\mu)(1 - \zeta_w)}{(1 + \bar{\beta}\mu)\zeta_w} \bar{A}_w \left[\frac{1}{1 - h/\mu}[\hat{c}_t - (h/\mu)\hat{c}_{t-1}] + \nu \hat{n}_t - \hat{w}_t + \frac{d\tau_t^n}{1 - \tau_n} + \frac{d\tau_t^c}{1 + \tau_c}] \right] \\
&- \frac{1 + \bar{\beta}\mu\iota_w}{1 + \bar{\beta}\mu}\hat{\pi}_t + \frac{\iota_w}{1 + \bar{\beta}\mu}\hat{\pi}_{t-1} + \frac{\bar{\beta}\mu}{1 + \bar{\beta}\mu}\mathbb{E}_t[\hat{\pi}_{t+1}] + \frac{\hat{\epsilon}_t^{\lambda,w}}{1 + \bar{\beta}\mu}.
\end{aligned} \tag{9.72}$$

From the consumption Euler equation (9.32):

$$\begin{aligned}
\mathbb{E}_t[\hat{\xi}_{t+1} - \hat{\xi}_t] &= \mathbb{E}_t\left((\sigma - 1)\bar{n}^{1+\nu}[\hat{n}_{t+1} - \hat{n}_t] - \frac{\sigma}{1 - h/\mu}\left[\hat{c}_{t+1}^{RA} - \left(1 + \frac{h}{\mu}\right) c_t^{RA} + \frac{h}{\mu}\hat{c}_{t+1}^{RA} \right] \right) \\
&= \frac{1}{1 - h/\mu}\mathbb{E}_t\big((\sigma - 1)\frac{\bar{n}^{1+\nu}[\bar{c}^{RA} - h/\mu\bar{c}^{RA}]}{\bar{c}^{RA}}[\hat{n}_{t+1} - \hat{n}_t]
\end{aligned}$$

---

[7] Here, the analogy with marginal costs holds only to a first order. Noting that common terms drop out the first order condition (9.39) and using (9.42) as well as $A_w \equiv [1 + \bar{\lambda}_w \hat{\eta}_w(1)]^{-1}$ linearizes as follows:

$$\begin{aligned}
0 &= \mathbb{E}_t\left(\sum_{s=0}^{\infty} (\zeta_w \mu \bar{\beta})^s \frac{\bar{n}}{\bar{\lambda}_w} \bar{w}^* \left(\big[\hat{w}_t^* + \sum_{l=1}^{s} (\iota_w \hat{\pi}_{t+l-1} - \hat{\pi}_{t+l})\big](1 + \bar{\lambda}\hat{\eta}_w(1)) - \bar{\lambda}_w \hat{\eta}_w(1)\hat{w}_{t+s} + \hat{w}_{t+s}^h + \hat{\bar{\epsilon}}_{t+s}^{\lambda,w} \right) \right) \\
&\propto \frac{1}{1 - \zeta_w \mu \bar{\beta}} A_w^{-1}[\hat{w}_t^* + \iota_w \hat{\pi}_t - \mathbb{E}_t(\hat{\pi}_{t+1})] \\
&+ \mathbb{E}_t\left(\sum_{s=0}^{\infty} (\zeta_w \mu \bar{\beta})^s \left(\big[A_w^{-1} \sum_{l=1}^{s-1} (\iota_w \hat{\pi}_{t+l} - \hat{\pi}_{t+l+1})\big](1 + \bar{\lambda}\hat{\eta}_w(1)) - [A_w^{-1} - 1]\hat{w}_{t+s} - \hat{w}_{t+s}^h - \hat{\bar{\epsilon}}_{t+s}^{\lambda,w} \right) \right) \\
&\propto \frac{1}{1 - \zeta_w \mu \bar{\beta}} A_w^{-1}[\hat{w}_t^* + \iota_w \hat{\pi}_t - \mathbb{E}_t(\hat{\pi}_{t+1}) - \zeta_w \mu \bar{\beta}\mathbb{E}_t(w_{t+1}^*)] - \hat{w}_t^h - \hat{\bar{\epsilon}}_t^{\lambda,w} - (1 - A_w^{-1})\hat{w}_t
\end{aligned}$$

Log-linearizing the law of motion for aggregate wages (9.41) around the symmetric balanced growth path yields:

$$\hat{w}_t^* = \frac{1}{1 - \zeta_w}[\hat{w}_t - \zeta_w \hat{w}_{t-1} - \zeta \iota_w \hat{\pi}_{t-1} + \zeta_w \hat{\pi}_t.$$

Substituting this equation into the above for $\hat{w}_t^*, \hat{w}_{t+1}^*$ and re-arranging yields (9.72).

$$- \sigma\left[\hat{c}^{RA}_{t+1} - \left(1+\frac{h}{\mu}\right)c^{RA}_t + \frac{h}{\mu}\hat{c}^{RA}_{t+1}\right]\Big)$$

$$= \frac{1}{1-h/\mu}\mathbb{E}_t\Big((\sigma-1)\frac{1}{1+\bar{\lambda}_w}\frac{1-\bar{\tau}^n}{1+\tau^c}\frac{\bar{w}\bar{n}}{\bar{c}^{RA}}[\hat{n}_{t+1}-\hat{n}_t]$$
$$- \sigma\left[\hat{c}^{RA}_{t+1} - \left(1+\frac{h}{\mu}\right)c^{RA}_t + \frac{h}{\mu}\hat{c}^{RA}_{t+1}\right]\Big),$$

where the last equality uses (9.42). Solving for current consumption growth:

$$\hat{c}^{RA}_t = \frac{1}{1+h/\mu}\mathbb{E}_t[\hat{c}^{RA}_{t+1}] + \frac{h/\mu}{1+h/\mu}\hat{c}^{RA}_{t-1} + \frac{1-h/\mu}{\sigma[1+h/\mu]}\mathbb{E}_t[\hat{\xi}_{t+1}-\hat{\xi}_t]$$
$$- \frac{[\sigma-1][\bar{w}\bar{n}/\bar{c}]}{\sigma[1+h/\mu]}\frac{1}{1+\lambda_w}\frac{1-\tau^n}{1+\tau^c}(\mathbb{E}_t[\hat{n}_{t+1}]-\hat{n}_t). \tag{9.73}$$

The remaining households' FOC linearize as:

$$\mathbb{E}_t[\hat{\xi}_{t+1}-\hat{\xi}_t] = -\hat{q}^b_t - \hat{R}_t + \mathbb{E}_t[\hat{\pi}_{t+1}], \tag{9.74a}$$

$$\hat{Q}_t = -\hat{q}^b_t - (\hat{R}_t - \mathbb{E}_t[\pi_{t+1}]) + \frac{1}{\bar{r}^k(1-\tau^k)+\delta\tau^k+1-\delta}\times$$
$$\times\left[(\bar{r}^k(1-\tau^k)+\delta\tau^k)\hat{q}^k_t - (\bar{r}^k-\delta)d\tau^k_{t+1} + \bar{r}^k(1-\tau^k)\mathbb{E}_t(\hat{r}^k_{t+1}) + (1-\delta)\mathbb{E}_t(\hat{Q}_{t+1})\right], \tag{9.74b}$$

$$\hat{x}_t = \frac{1}{1+\bar{\beta}\mu}\left[\hat{x}_{t-1} + \bar{\beta}\mu\mathbb{E}_t(\hat{x}_{t+1}) + \frac{1}{\mu^2 S''(\mu)}[\hat{Q}_t + \hat{q}^x_t]\right], \tag{9.74c}$$

$$\hat{u}_t = \frac{a'(1)}{a''(1)}\hat{r}^k_t \equiv \frac{1-\psi_u}{\psi_u}\hat{r}^k_t. \tag{9.74d}$$

For the credit constrained households, (9.35) implies the following linear consumption process: consumption evolves as

$$\hat{c}^{RoT}_t = \frac{1}{1+\tau^c}\left(\frac{\bar{s}^{RoT}}{\bar{c}^{RoT}}\hat{s}_t + \frac{\bar{w}\bar{n}}{\bar{c}^{RoT}}[(1-\tau^n)(\hat{w}_t+\hat{n}_t) - d\tau^n_t] - d\tau^c_t + \frac{\bar{y}}{\bar{c}^{RoT}}\frac{d\Pi^p_t}{\bar{y}}\right), \tag{9.75}$$

where the change in profits is given by:

$$\frac{d\Pi^p_t}{\bar{y}} = \frac{1}{1+\lambda_p}\hat{y}_t - \widehat{mc}_t.$$

9.6.3 Government

The financing need evolves as:

$$\frac{dd_t}{\bar{y}} = \frac{1}{\mu}\Bigg[\mu[\hat{g}_t^a + \hat{g}^s] + \mu\frac{\bar{s}}{\bar{y}}\hat{s}_t^{exog} + \frac{\bar{b}}{\bar{y}}\frac{\hat{b}_{t-1} - \hat{\pi}_t}{\bar{\pi}} - \mu\tau^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\Big(\hat{w}_t + \hat{n}_t\Big)$$
$$- \mu\tau_c\frac{\bar{c}}{\bar{y}}\hat{c}_t - \tau^k[\bar{r}^k r_t^k + (r_t^k - \delta)\hat{k}_{t-1}^p]\mu\frac{\bar{k}}{\bar{y}}\Bigg]. \tag{9.76}$$

In the benchmark case of distortionary labor taxes, Labor tax rates evolve according to (9.45), which is linearized as:

$$\bar{\tau}^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\left[\frac{d\tau_t^n}{\tau_n}\right] + \hat{\epsilon}_t^\tau = \psi_\tau\frac{dd_t}{\bar{y}}$$
$$= \frac{\psi_\tau}{\mu}\Bigg[\mu[\hat{g}_t^a + \hat{g}^s] + \mu\frac{\bar{s}}{\bar{y}}\hat{s}_t^{exog} + \frac{\bar{b}}{\bar{y}}\frac{\hat{b}_{t-1} - \hat{\pi}_t}{\bar{\pi}} - \mu\tau^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\Big(\hat{w}_t + \hat{n}_t\Big)$$
$$- \mu\tau_c\frac{\bar{c}}{\bar{y}}\hat{c}_t - \tau^k[\bar{r}^k r_t^k + (r_t^k - \delta)\hat{k}_{t-1}^p]\mu\frac{\bar{k}}{\bar{y}}\Bigg]. \tag{9.77}$$

In general, tax rates, or endogenous transfers satisfy from (9.46):

$$\bar{\tau}^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\left[\frac{d\tau_t^n}{\tau_n}\right] + \tau^c\frac{\bar{c}}{\bar{y}}\frac{d\tau_t^c}{\tau^c} + \tau^k\frac{[\bar{r}^k - \delta]\bar{k}}{\bar{y}}\frac{d\tau_t^k}{\tau^k} - \frac{\bar{s}}{\bar{y}}\hat{s}_t^{endog} + \hat{\epsilon}_t^\tau = \psi_\tau\frac{dd_t}{\bar{y}} \tag{9.78}$$

Note how the bond shock is treated here! Might want to change it for estimation etc. purposes. Check!!! Debt holdings are determined from the budget constraint (9.44):

$$\frac{1}{\bar{R}}\frac{\bar{b}}{\bar{y}}[\hat{b}_t - \hat{R}_t - \hat{q}_t^b] = (1-\psi_\tau)\frac{dd_t}{\bar{y}} - \bar{\tau}^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\left[\frac{d\tau_t^n}{\tau_n}\right] - \tau^c\frac{\bar{c}}{\bar{y}}\frac{d\tau_t^c}{\tau^c} - \tau^k\frac{[\bar{r}^k - \delta]\bar{k}}{\bar{y}}\frac{d\tau_t^k}{\tau^k} + \frac{\bar{s}}{\bar{y}}\hat{s}_t^{endog} - \hat{\epsilon}_t^\tau \tag{9.79}$$

The linearized counterpart to the law of motion for government capital (9.47) is given by:

$$\hat{k}^g = \left(1 - \frac{\bar{x}^g}{\bar{k}^g}\right)\hat{k}_{t-1}^g + \frac{\bar{x}^g}{\bar{k}^g}\hat{q}_t^{x,g} + \frac{\bar{x}^g}{\bar{k}^g}[\hat{x}_t^g + \hat{\epsilon}_t^{xg}], \tag{9.80}$$

where $u_t^{x,g} \equiv \frac{\tilde{u}_t^{x,g}}{\bar{x}^g}$.

The marginal product of government capital (9.49) is approximated by

$$\hat{r}_t^g = \frac{\bar{y}}{\bar{y} + \Phi}\hat{y}_t - \hat{k}_{t-1}^g \tag{9.81}$$

The shadow price of government capital (9.48b) has the following linear approximation:

$$\hat{Q}_t^g = -(\hat{R}_t + \hat{q}_t^b - \mathbb{E}_t[\pi_{t+1}]) + \frac{1}{\bar{r}^g + 1 - \delta}[\bar{r}^g \mathbb{E}_t(\hat{r}_{t+1}^g) + (1-\delta)\mathbb{E}_t(\hat{Q}_{t+1}^g)], \tag{9.82}$$

The Euler equation for government investment (9.48a) is approximated as:

$$\hat{x}_t^g = \frac{1}{1+\bar{\beta}\mu}\left[\hat{x}_{t-1} + u_{t-1}^{xg} + \bar{\beta}\mu\mathbb{E}_t([\hat{x}_{t+1}^g + u_{t+1}^{xg}]) + \frac{1}{\mu^2 S_g''(\mu)}[\hat{Q}_t^g + \hat{q}_t^{x,g}]\right] - u_t^{xg} \tag{9.83}$$

The monetary policy rule (9.52) is approximated by:

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)[\psi_1 \hat{\pi}_t + \psi_2(\hat{y}_t - \hat{y}_t^f)] + \psi_3 \Delta(\hat{y}_t - \hat{y}_t^f) + \hat{\epsilon}_t^r \tag{9.84}$$

9.6.4 Exogenous processes

The shock processes (9.57) are linearized as

$$\hat{\epsilon}_t^a = \rho_a \hat{\epsilon}_{t-1}^a + u_t^a, \tag{9.85a}$$
$$\hat{\epsilon}_t^r = \rho_r \hat{\epsilon}_{t-1}^r + u_t^r, \tag{9.85b}$$
$$\hat{g}_t = \hat{g}_t^a + \tilde{u}_t^g, \tag{9.85c}$$
$$\hat{g}_t^a = \rho_g \hat{g}_{t-1}^a + \sigma_{ga} u_t^a + u_t^g, \tag{9.85d}$$
$$\hat{s}_t = \tilde{u}_t^s, \tag{9.85e}$$
$$\hat{\epsilon}_t^\tau = \rho_\tau \hat{\epsilon}_{t-1}^\tau + u_t^\tau, \tag{9.85f}$$
$$\hat{\hat{\epsilon}}_t^{\lambda,p} = \rho_{\lambda,p} \hat{\hat{\epsilon}}_{t-1}^{\lambda,p} + u_t^{\lambda,p} - \theta_{\lambda,p} u_{t-1}^{\lambda,p}, \tag{9.85g}$$
$$\hat{\hat{\epsilon}}_t^{\lambda,w} = \rho_{\lambda,w} \hat{\hat{\epsilon}}_{t-1}^{\lambda,w} + u_t^{\lambda,w} - \theta_{\lambda,w} u_{t-1}^{\lambda,w}, \tag{9.85h}$$
$$\hat{q}_t^b = \rho_b \hat{q}_{t-1}^b + u_t^b, \tag{9.85i}$$

$$\hat{q}_t^k = \rho_k \hat{q}_{t-1}^k + u_t^k, \tag{9.85j}$$

$$\hat{q}_t^x = \rho_x \hat{q}_{t-1}^x + u_t^x, \tag{9.85k}$$

$$\hat{q}_t^{x,g} = \rho_{x,g} \hat{q}_{t-1}^{x,g} + u_t^{x,g}. \tag{9.85l}$$

9.6.5 Aggregation

Aggregate consumption (9.59) and transfers (9.61) are linearized as

$$\hat{c}_t = (1-\phi)\frac{\bar{c}^{RA}}{\bar{c}}\hat{c}_t^{RA} + \phi\frac{\bar{c}^{RoT}}{\bar{c}}\hat{c}_t^{RoT}, \tag{9.86}$$

$$\hat{s}_t = (1-\phi)\frac{\bar{s}^{RA}}{\bar{s}}\hat{s}_t^{RA} + \phi\frac{\bar{s}^{RoT}}{\bar{s}}\hat{s}_t^{RoT}. \tag{9.87}$$

9.6.6 Market Clearing

Goods market clearing:

$$\hat{y}_t = \frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{x}}{\bar{y}}\hat{x}_t + \frac{\bar{x}^g}{\bar{y}}\hat{x}_t^g + \hat{g}_t + \frac{\bar{r}^k\bar{k}}{\bar{y}}\hat{u}_t. \tag{9.88}$$

9.6.7 Solution

In addition to the exogenous processes in (9.85), the economy with frictions is reduced to 21 variables, whereas the flexible economy is characterized by 19 variables only, given perfectly flexible prices and wages. Table 13 on the following page lists the remaining variables and the corresponding equations. For the flexible economy, all variables other than those with an “n/a” entry have an ^{f} superscript. The markup shock processes affect only the economy with frictions. Table 14 on page 83 lists the steady state relationships which enter the linearized equations.

Variable	Economy with frictions	Economy without frictions
$\hat{c}$	(9.86)	(9.86)
$\hat{c}^{RA}$	(9.73)	(9.73)
$\hat{c}^{RoT}$	(9.75)	(9.75)
$\hat{x}$	(9.74a) in (9.74c)	(9.74c), (9.74a)
$\hat{k}^p$	(9.68)	(9.68)
$\hat{k}$	(9.69)	(9.69)
$\hat{u}$	(9.74d)	(9.74d)
$\hat{Q}$	(9.74a) in (9.74b)	(9.74b), (9.74a)
$\hat{r}^k$	(9.64)	(9.64)
$\hat{x}^g$	(9.74a) in (9.83)	(9.83), (9.74a)
$\hat{k}^g$	(9.80)	(9.80)
$\hat{Q}^g$	(9.74a) in (9.82)	(9.82), (9.74a)
$\hat{r}^g$	(9.81)	(9.81)
$d\tau^n, d\tau^c, d\tau^k, \hat{s}^{endo}$	one variable according to (9.78) with (9.76)	(9.78) with (9.76)
	other three variables = 0	other three variables = 0
$\hat{b}$	(9.79)	(9.79)
$\hat{R}$	(9.84)	indirectly via (9.66)
$\hat{\pi}$	(9.67)	=0
$\widehat{mc}$	(9.65)	=0
$\hat{w}$	(9.72)	(9.71)
$\hat{y}$	(9.88)	(9.88)
$\hat{n}$	(9.63)	(9.63)

Table 13: Unknowns and equations

Constant	Equation	Expression
$\frac{\bar{c}}{\bar{y}}$	(9.62)	$1 - \frac{\bar{x}}{\bar{y}} - \frac{\bar{x}^g}{\bar{y}} - g$
$\frac{\bar{c}^{RA}}{\bar{y}}$	(9.60)	$\frac{\bar{c} - \phi \bar{c}^{RoT}}{\bar{y}(1-\phi)}$
$\frac{\bar{c}^{RoT}}{\bar{y}}$	(9.36)	$\frac{\bar{s}^{RoT} + (1-\tau^n)\bar{w}\bar{n}}{\bar{y}(1+\tau^c)}$
$\frac{\bar{x}}{\bar{k}^p}$	(9.30)	$1 - \frac{1-\delta}{\mu}$
$\frac{\bar{x}}{\bar{k}}$	(9.30)	$\mu - (1-\delta)$
$\frac{\bar{k}}{\bar{y}}$	(9.8)	$\left(\frac{\bar{y}+\Phi}{\bar{y}}\right)^{\frac{1}{1-\zeta}} \left(\frac{\bar{k}^g}{\bar{y}}\right)^{\frac{-\zeta}{1-\zeta}} \left(\frac{\bar{k}}{\bar{n}}\right)^{1-\alpha}$
$\bar{u}$	normalization	$a'^{-1}(\bar{r}^k)$
$\bar{\beta}$	definition	$\beta \mu^{-1}$
$\bar{r}^k$	(9.33c)	$\frac{\bar{\beta}^{-1} - \delta\tau^k - (1-\delta)}{1-\tau^k}$
$\frac{\bar{k}^g}{\bar{y}}$	(9.47)	$\left(1 - \frac{1-\delta}{\mu}\right)^{-1} \frac{\bar{x}^g}{\bar{y}}$
ζ	(9.51)	$\frac{\bar{y}}{\bar{y}+\Phi} \frac{\bar{r}^g}{1-(1-\delta)/\mu} \frac{\bar{x}}{\bar{y}}$
$\bar{r}^g$	(9.50)	$\bar{\beta}^{-1} - (1-\delta)$
$\bar{R}$	(9.33b)	$\bar{\beta}^{-1}\bar{\pi}$
$\overline{mc}$	(9.16)	$(1+\bar{\lambda}_p)^{-1}$
$\bar{\lambda}_p$	(9.18)	$\frac{\Phi}{\bar{y}}$
$\bar{w}$	(9.11)	$\frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{(1+\lambda_w)^{\frac{1}{(1-\zeta)(1-\alpha)}}} \frac{\left(\frac{\bar{k}^g}{\bar{y}}\right)^{\frac{\zeta}{(1-\zeta)(1-\alpha)}}}{\bar{r^k}^{\frac{\alpha}{1-\alpha}}}$
$\frac{\bar{w}\bar{n}}{\bar{y}}$	$[n_t(i)], [K_t(i)]$,(9.16),(9.18)	$1 - \bar{r}^k \frac{\bar{k}}{\bar{y}}$
$\frac{\bar{k}}{\bar{n}}$	(9.9)	$\frac{\alpha}{1-\alpha}\frac{\bar{w}}{\bar{r}^k}$

Table 14: Steady state relationships

9.7 Measurement equations

For the estimation of the model, the following measurement equations are appended to the model:

$$\Delta Y_t = 100(\hat{y}_t - \hat{y}_{t-1}) + 100(\mu - 1), \tag{9.89a}$$

$$\Delta C_t = 100(\hat{c}_t - \hat{c}_{t-1}) + 100(\mu - 1), \tag{9.89b}$$

$$\Delta X_t = 100(\hat{x}_t - \hat{x}_{t-1}) + 100(\mu - 1), \tag{9.89c}$$

$$\Delta X_t^g = 100(\hat{x}_t^g - \hat{x}_{t-1}^g) + 100(\mu - 1), \tag{9.89d}$$

$$\Delta \frac{W_t}{P_t} = 100(\hat{w}_t - \hat{w}_{t-1}) + 100(\mu - 1), \tag{9.89e}$$

$$\hat{\pi}_t^{obs} = 100\hat{\pi}_t + 100(\bar{\pi} - 1), \tag{9.89f}$$

$$\hat{R}_t^{obs} = 100\hat{R}_t + 100(\bar{R} - 1), \tag{9.89g}$$

$$\hat{q}_t^{k,obs} = 100\hat{q}_t^k + \bar{\hat{q}}^{k,obs}, \tag{9.89h}$$

$$\hat{n}_t^{obs} = 100\hat{n}_t + \bar{\hat{n}}^{obs}, \tag{9.89i}$$

$$\hat{b}_t^{obs} = 100\hat{b}_t + \bar{\hat{b}}^{obs}. \tag{9.89j}$$

The constants give the inflation rate $\bar{\pi}$ along the balanced growth path and the trend growth rates. $100(\mu - 1)$ represents the deterministic net trend growth imposed on the data,. Note that apart from the trend growth rate and the constant nominal interest rate, the discount factor can be backed out of the constants:

$$\beta = \frac{\bar{\pi}}{\bar{R}} \mu^{\sigma}.$$

The constant terms in the measurement equation are necessary even if the data is demeaned for the particular observation sample because the allocation in the flexible economy cannot be attained in the economy with frictions. Given a non-zero output gap, also other variables will deviate from zero. To see why notice that for the allocations to be the same in both the economy with frictions and the its frictionless counterpart required that the Calvo constraints on price and wage setting were slack – otherwise the equilibrium allocations would differ from that in the flexible economy. Slack

Calvo constraints in turn required that aggregate prices and wages were constant, which implied a constant real wage. Finally, a constant real wage would be inconsistent with the allocation in the flexible economy.

9.8 Welfare implications

To evaluate welfare implications, we approximate the compensating variation in terms of quarterly consumption of each type of agent separately as well as the population weighted average.

Independent of whether a household is constrained or not, equation (9.24) gives the preferences of the household. Using the log-linearized model solution around the deterministic balanced growth path, the lifetime utility of any time-path of consumption and hours worked can be computed as:

$$\begin{aligned}
U_t(\{\hat{c}_{t+s}, \hat{n}_{t+s}\}) &= \sum_{s=0}^{\infty} \beta^s \left[\frac{(\mu^{1-\sigma})^{t+s}}{1-\sigma} \left(\bar{c} \exp[\hat{c}_{t+s}] - \frac{h}{\mu} \bar{c} \exp[\hat{c}_{t+s-1}] \right)^{1-\sigma} \right] \\
&\qquad \times \exp\left[\frac{\sigma - 1}{1+\nu} (\bar{n} \exp[\hat{n}_{t+s}])^{1+\nu} \right] \\
&= (\mu^{1-\sigma})^t \sum_{s=0}^{\infty} [\beta\mu^{1-\sigma}]^s \left[\frac{\bar{c}^{1-\sigma}}{1-\sigma} \left(\exp[\hat{c}_{t+s}] - \frac{h}{\mu} \exp[\hat{c}_{t+s-1}] \right)^{1-\sigma} \right] \\
&\qquad \times \exp\left[-\frac{\bar{n}^{1+\nu}}{1+\nu} \exp[(1+\nu)\hat{n}_{t+s}] \right]^{1-\sigma} \\
&= (\mu^{1-\sigma})^t \frac{\bar{c}^{1-\sigma}}{1-\sigma} \sum_{s=0}^{\infty} [\beta\mu^{1-\sigma}]^s \left[\left(e^{\hat{c}_{t+s}} - \frac{h}{\mu} e^{\hat{c}_{t+s-1}} \right) \exp\left[-\frac{\bar{n}^{1+\nu}}{1+\nu} \exp[(1+\nu)\hat{n}_{t+s}] \right] \right]^{1-\sigma}.
\end{aligned}$$

Now we can compute the compensating variation between to paths of consumption and leisure, with and without the fiscal stimulus as:

$$\Gamma = \left[\frac{\sum_{s=0}^{\infty} [\beta\mu^{1-\sigma}]^s \left(e^{\hat{c}_{t+s}^{ARRA}} - \frac{h}{\mu} e^{\hat{c}_{t+s-1}^{ARRA}} \right) \exp\left[-\frac{\bar{n}^{1+\nu}}{1+\nu} \left(\exp[(1+\nu)\hat{n}_{t+s}^{ARRA}] - 1 \right) \right] \Big)^{1-\sigma}}{\sum_{s=0}^{\infty} [\beta\mu^{1-\sigma}]^s \left(e^{\hat{c}_{t+s}^{wo}} - \frac{h}{\mu} e^{\hat{c}_{t+s-1}^{wo}} \right) \exp\left[-\frac{\bar{n}^{1+\nu}}{1+\nu} \left(\exp[(1+\nu)\hat{n}_{t+s}^{wo}] - 1 \right) \right] \Big)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}} - 1. \tag{9.90}$$

An individual with discount factor β would be willing to give up a fraction Γ of consumption in each period to live in an otherwise identical work with the fiscal stimulus in place.

For large s the deviations from the balanced growth path are numerically indistinguishable from zero. However, since $\beta\mu^{1-\sigma}$ is in practice close to unity, even for $s = 1,000$, the infinite sum has not converged. We therefore approximate:

$$\sum_{s=0}^{\infty}[\beta\mu^{1-\sigma}]^s \left[\left(e^{\hat{c}_{t+s}} - \frac{h}{\mu}e^{\hat{c}_{t+s-1}}\right)\exp\left[-\frac{\bar{n}^{1+\nu}}{1+\nu}\left(\exp[(1+\nu)\hat{n}_{t+s}] - 1\right)\right]\right]^{1-\sigma}$$
$$\approx \sum_{s=0}^{T}[\beta\mu^{1-\sigma}]^s \left[\left(e^{\hat{c}_{t+s}} - \frac{h}{\mu}e^{\hat{c}_{t+s-1}}\right)\exp\left[-\frac{\bar{n}^{1+\nu}}{1+\nu}\left(\exp[(1+\nu)\hat{n}_{t+s}] - 1\right)\right]\right]^{1-\sigma}$$
$$+ \frac{[\beta\mu^{1-\sigma}]^{T+1}}{1-\beta\mu^{1-\sigma}]^s}(1-h/\mu)^{1-\sigma},$$

for some large T. In practice, we use $T = 1000$ but checked the results for $T = 5,000$.

To obtain $\bar{n}^{1+\nu}$, multiply equation (9.42) by $\bar{n}$ and divide by $\bar{y}$. This shows that $\bar{n}^{1+\nu} = \frac{\bar{w}\bar{n}}{\bar{y}}\frac{1}{(1+\lambda^w)}\frac{1}{\bar{c}^{RA}/\bar{y}}\frac{1}{1-\frac{h}{\mu}}\frac{1-\bar{\tau}^n}{1+\tau^c}$, which is in terms of the constants in table 14.