Appendix to "Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria"

A Solving the Two-Equation Model

The model is characterized by the nonlinear difference equation

$$\mathbb{E}_t[\pi_{t+1}] = \max \left\{ \frac{1}{r},\ \pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi} \exp[\epsilon_t] \right\}. \tag{A.1}$$

We assume that $r\pi_* \geq 1$ and $\psi > 1$.

The Targeted-Inflation Equilibrium and Deflation Equilibrium. Consider a solution to (A.1) that takes the following form

$$\pi_t = \pi_* \gamma \exp[\lambda \epsilon_t]. \tag{A.2}$$

We now determine values of γ and λ such that (A.1) is satisfied. We begin by calculating the following expectation

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}] &= \pi_* \gamma \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp[\lambda\epsilon] \exp\left[-\frac{1}{2\sigma^2} \epsilon^2 \right] d\epsilon \\
&= \pi_* \gamma \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{1}{2} \lambda^2 \sigma^2 \right] \int \exp\left[-\frac{1}{2\sigma^2} (\epsilon - \lambda\sigma^2)^2 \right] d\epsilon \\
&= \pi_* \gamma \exp\left[\frac{1}{2} \lambda^2 \sigma^2 \right].
\end{aligned}$$

Combining this expression with (A.1) yields

$$\gamma \exp[\lambda^2 \sigma^2 / 2] = \max \left\{ \frac{1}{r\pi_*},\ \gamma^{\psi} \exp[(\psi\lambda + 1)\epsilon_t] \right\}. \tag{A.3}$$

By choosing $\lambda = -1/\psi$, we ensure that the right-hand side of (A.3) is always constant. Thus, (A.3) reduces to

$$\gamma \exp[\sigma^2 / (2\psi^2)] = \max \left\{ \frac{1}{r\pi_*},\ \gamma^{\psi} \right\} \tag{A.4}$$

Depending on whether the nominal interest rate is at the ZLB ($R_t = 1$) or not, we obtain two solutions for γ by equating the left-hand-side of (A.4) with either the first or the second term in the max operator:

$$\gamma_D = \frac{1}{r\pi_*} \exp\left[-\frac{\sigma^2}{2\psi^2} \right] \quad \text{and} \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi - 1)\psi^2} \right]. \tag{A.5}$$

The derivation is completed by noting that

$$\begin{aligned}
\gamma_D^\psi &= \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi}\right] \le \frac{1}{r\pi_*} \\
\gamma_*^\psi &= \exp\left[\frac{\sigma^2}{2(\psi-1)\psi}\right] \ge 1 \ge \frac{1}{r\pi_*}.
\end{aligned}$$

A Sunspot Equilibrium. Let $s_t \in \{0,1\}$ denote the Markov-switching sunspot process. Assume the system is in the targeted-inflation regime if $s_t = 1$ and that it is in the deflation regime if $s_t = 0$ (the 0 is used to indicate that the system is near the ZLB). The probabilities of staying in state 0 and 1, respectively, are denoted by ψ_{00} and ψ_{11}. We conjecture that the inflation dynamics follow the process

$$\pi_t^{(s)} = \pi_*\gamma(s_t)\exp[-\epsilon_t/\psi] \tag{A.6}$$

In this case condition (A.4) turns into

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}|s_t=0]/\pi_* &= \big(\psi_{00}\gamma(0) + (1-\psi_{00})\gamma(1)\big)\exp[\sigma^2/(2\psi^2)] = \frac{1}{r\pi_*} \\
\mathbb{E}_t[\pi_{t+1}|s_t=1]/\pi_* &= \big(\psi_{11}\gamma(1) + (1-\psi_{11})\gamma(0)\big)\exp[\sigma^2/(2\psi^2)] = [\gamma(1)]^\psi.
\end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the Markov-transition probabilities ψ_{00} and ψ_{11}. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^\psi \le \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^\psi \ge \frac{1}{r\pi_*}.$$

Sunspot Shock Correlated with Fundamentals. As before, let $s_t \in \{0,1\}$ be a Markov-switching sunspot process. However, now assume that a state transition is triggered by certain realizations of the monetary policy shock ϵ_t. In particular, if $s_t = 0$, then suppose $s_{t+1} = 0$ whenever $\epsilon_{t+1} \le \underline{\epsilon}_0$, such that

$$\psi_{00} = \Phi(\underline{\epsilon}_0),$$

where $\Phi(\cdot)$ is the cumulative density function of a $N(0,1)$. Likewise, if $s_t = 1$, then let $s_{t+1} = 0$ whenever $\epsilon_{t+1} > \underline{\epsilon}_0$, such that

$$\psi_{11} = 1 - \Phi(\underline{\epsilon}_1).$$

To find the constants $\gamma(0)$ and $\gamma(1)$, we need to evaluate

$$\begin{aligned} & \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\underline{\epsilon}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2 \right] d\epsilon \\ = & \ \mathbb{P}\left\{ \frac{\epsilon + \sigma^2/\psi}{\sigma} \leq \frac{\underline{\epsilon} + \sigma^2/\psi}{\sigma} \right\} = \Phi\left(\frac{\underline{\epsilon} + \sigma^2/\psi}{\sigma} \right). \end{aligned}$$

Thus, condition (A.4) turns into

$$\begin{aligned} \frac{1}{r\pi_*} &= \left[\gamma(0)\Phi(\underline{\epsilon}_0)\Phi\left(\frac{\underline{\epsilon}_0 + \sigma^2/\psi}{\sigma} \right) + \gamma(1)(1 - \Phi(\underline{\epsilon}_0)) \left(1 - \Phi\left(\frac{\underline{\epsilon}_0 + \sigma^2/\psi}{\sigma} \right) \right) \right] \exp[\sigma^2/(2\psi^2)] \\ \gamma^\psi(1) &= \left[\gamma(1)(1 - \Phi(\underline{\epsilon}_1)) \left(1 - \Phi\left(\frac{\underline{\epsilon}_1 + \sigma^2/\psi}{\sigma} \right) \right) + \gamma(0)\Phi(\underline{\epsilon}_1)\Phi\left(\frac{\underline{\epsilon}_1 + \sigma^2/\psi}{\sigma} \right) \right] \exp[\sigma^2/(2\psi^2)]. \end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the thresholds $\underline{\epsilon}_0$ and $\underline{\epsilon}_1$. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^\psi \leq \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^\psi \geq \frac{1}{r\pi_*}.$$

Benhabib, Schmitt-Grohé, and Uribe (2001a) Dynamics. BSGU constructed equilibria in which the economy transitioned from the targeted-inflation equilibrium to the deflation equilibrium. Consider the following law of motion for inflation

$$\pi_t^{(BGSU)} = \pi_* \gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0} \big]. \tag{A.7}$$

Here, γ_* was defined in (A.5) and $-t_0$ can be viewed as the initialization period for the inflation process. We need to verify that $\pi_t^{(BGSU)}$ satisfies (A.1). From the derivations that lead to (A.4) we deduce that

$$\gamma_* \mathbb{E}_{t+1}\big[\exp[-\epsilon_{t+1}/\psi] \big] = \gamma_*^\psi.$$

Since

$$\exp\big[-\psi^{t+1-t_0} \big] = \big(\exp\big[-\psi^{t-t_0} \big] \big)^\psi,$$

we deduce that the law of motion for $\pi_t^{(BGSU)}$ in (A.7) satisfies the relationship

$$\mathbb{E}_t[\pi_{t+1}] = \pi_* \left(\frac{\pi_t}{\pi_*} \right)^\psi \exp[\epsilon_t].$$

Moreover, since $\psi > 1$, the term $\exp\big[-\psi^{t-t_0}\psi \big] \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, the economy will move away from the targeted-inflation equilibrium and at some suitably defined t_* reach the deflation equilibrium and remain there permanently. Overall the inflation dynamics take the form

$$\pi_t = \pi_* \begin{cases} \gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0}\big] & \text{if } t \le t_* \\ \gamma_D \exp[-\epsilon_t/\psi] & \text{otherwise} \end{cases}, \tag{A.8}$$

where γ_* and γ_D were defined in (A.5).

Alternative Deflation Equilibria. Around the deflation steady state, the system is locally indeterminate. This suggests that we can construct alternative solutions to (A.1). Consider the following conjecture for inflation

$$\pi_t = \pi_* \gamma \min\big\{ \exp[-c/\psi], \exp[-\epsilon/\psi] \big\}, \tag{A.9}$$

where c is a cutoff value. The intuition for this solution is the following. Large positive shocks ϵ that could push the nominal interest rate above one, are offset by downward movements in inflation. Negative shocks do not need to be offset because they push the desired gross interest rate below one, and the max operator in the policy rule keeps the interest rate at one. Formally, we can compute the expected value of inflation as follows:

$$\begin{aligned} \mathbb{E}_t[\pi_{t+1}] &= \pi_* \gamma \Bigg[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{c} \exp[-c/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right] d\epsilon \\ &\quad \frac{1}{\sqrt{2\pi\sigma^2}} \int_{c}^{\infty} \exp[-\epsilon/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right] d\epsilon \\ &= \pi_* \gamma \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \int_c^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2\right] d\epsilon \right] \\ &= \pi_* \gamma \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi}\right)\right) \right] \end{aligned} \tag{A.10}$$

Here $\Phi(\cdot)$ denotes the cdf of a standard Normal random variable. Now define

$$f(c,\psi,\sigma) = \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi}\right)\right) \right].$$

Then another solution for which interest rates stay at the ZLB is given by

$$\bar{\gamma} = \frac{1}{r_* \pi_* f(c,\psi,\sigma)}$$

It can be verified that for c small enough, the condition

$$\frac{1}{r_* \pi_*} \ge \bar{\gamma}^{\psi} \min\left\{ \exp[-c+\epsilon],\, 1 \right\}$$

is satisfied.

B Model Solution

The equilibrium conditions (in terms of detrended variables, i.e., $c_t = C_t/A_t$ and $y_t = Y_t/A_t$) take the form

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \quad \text{(A.11)}$$

$$\begin{aligned} 1 &= \frac{1}{\nu}(1 - c_t^\tau) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] \quad \text{(A.12)} \\ &\quad - \phi\beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \end{aligned}$$

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t \quad \text{(A.13)}$$

$$R_t = \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. \quad \text{(A.14)}$$

B.1 Approximation Near the Targeted-Inflation Steady State

Steady State. Steady-state inflation equals π_*. Let $\lambda = \nu(1-\beta)$, then

$$\begin{aligned} r &= \gamma/\beta \\ R_* &= r\pi_* \\ c_* &= \left[1 - v - \frac{\phi}{2}(1 - 2\lambda) \left(\pi_* - \frac{1-\lambda}{1-2\lambda}\bar{\pi} \right)^2 + \frac{\phi}{2} \frac{\lambda^2}{1-2\lambda} \bar{\pi}^2 \right]^{1/\tau} \\ y_* &= \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_* - \bar{\pi})^2 \right]}. \end{aligned}$$

Log-Linearization. We omit the hats from variables that capture deviations from the targeted-inflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price setting equation (A.12) takes the form

$$\begin{aligned} 0 &= -\frac{\tau c_*^\tau}{\nu} c_t + \phi\pi_* \left[\left(1 - \frac{1}{2\nu} \right) \pi_* + \frac{\bar{\pi}}{2\nu} \right] \pi_t + \phi\pi_*(\pi_* - \bar{\pi}) \left(1 - \frac{1}{2\nu} \right) \pi_t \\ &\quad - \phi\beta\pi_*(\pi_* - \bar{\pi}) \Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}] \Big) - \phi\beta\pi_*^2 \mathbb{E}_t[\pi_{t+1}]. \end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\pi_* - \bar{\pi})^2} g_t - \frac{\phi\pi_*(\pi_* - \bar{\pi})}{1/g_* - \phi(\pi_* - \bar{\pi})^2}\pi_t$$

Finally, the monetary policy rule becomes

$$R_t = \max\left\{ -\ln(r\pi_*),\ (1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R\epsilon_{R,t}\right\}.$$

Approximate Piecewise-Linear Solution in Special Case. To simplify the exposition, we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and $\rho_g = 0$. We obtain the system

$$\begin{aligned} R_t &= \max\left\{ -\ln(r\pi_*),\ \psi\pi_t + \sigma_R\epsilon_{R,t}\right\} \\ c_t &= \mathbb{E}_t[c_{t+1}] - (R_t - \mathbb{E}_t[\pi_{t+1}]) \\ \pi_t &= \beta\mathbb{E}_t[\pi_{t+1}] + \kappa c_t. \end{aligned} \tag{A.15}$$

It is well known that if the shocks are small enough such that the ZLB is non-binding, the linearized system has a unique stable solution for $\psi > 1$. Since the exogenous shocks are *iid* and the simplified system has no endogenous propagation mechanism, consumption, output, inflation, and interest rates will also be *iid* and can be expressed as a function of $\epsilon_{R,t}$. In turn, the conditional expectations of inflation and consumption equal their unconditional means, which we denote by μ_π and μ_c, respectively.

The Euler equation in (A.15) simplifies to the static relationship

$$c_t = -R_t + \mu_c + \mu_\pi. \tag{A.16}$$

Similarly, the Phillips curve in (A.15) becomes

$$\pi_t = \kappa c_t + \beta\mu_\pi. \tag{A.17}$$

Combining (A.16) and (A.17) yields

$$\pi_t = -\kappa R_t + (\kappa + \beta)\mu_\pi + \kappa\mu_c. \tag{A.18}$$

We now can use (A.18) to eliminate inflation from the monetary policy rule:

$$R_t = \max\left\{ -\ln(r\pi_*),\ -\kappa\psi R_t + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right\} \tag{A.19}$$

Define

$$R_t^{(1)} = -\ln(r\pi_*) \quad \text{and} \quad R_t^{(2)} = \frac{1}{1+\kappa\psi}\bigg[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\bigg].$$

Let $\bar{\epsilon}_{R,t}$ be the value of the monetary policy shock for which $R_t = -\ln(r\pi_*)$ and the two terms in the max operator of (A.19) are equal

$$\sigma_R\bar{\epsilon}_{R,t} = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c.$$

To complete the derivation of the equilibrium interest rate, it is useful to distinguish the following two cases. Case (i): suppose that $\epsilon_{R,t} < \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(1)}$ is consistent with (A.19). If the monetary policy shock is less than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} < -(1+\kappa\psi)\ln(r\pi_*).$$

Thus,

$$-\kappa\psi R_t^{(1)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} < -\kappa\psi R_t^{(1)} - (1+\kappa\psi)\ln(r\pi_*) = -\ln(r\pi_*),$$

which confirms that (A.19) is satisfied.

Case (ii): suppose that $\epsilon_{R,t} > \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(2)}$ is consistent with (A.19). If the monetary policy shock is greater than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} > -(1+\kappa\psi)\ln(r\pi_*).$$

In turn,

$$\begin{aligned}
&-\kappa\psi R_t^{(2)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= -\frac{\kappa\psi}{1+\kappa\psi}\bigg[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\bigg] + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= \frac{1}{1+\kappa\psi}\bigg[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\bigg] \\
&> -\ln(r\pi_*),
\end{aligned}$$

which confirms that (A.19) is satisfied.

We can now deduce that

$$R_t(\epsilon_{R,t}) = \max\left\{-\ln(r\pi_*),\ \frac{1}{1+\kappa\psi}\bigg[\psi(\kappa+\beta)\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\bigg]\right\}. \tag{A.20}$$

Combining (A.16) and (A.20) yields equilibrium consumption

$$c_t(\epsilon_{R,t}) = \begin{cases} \frac{1}{1+\kappa\psi}\Big[(1-\psi\beta)\mu_\pi + \mu_c - \sigma_R\epsilon_{R,t}\Big] & \text{if } R_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c + \mu_\pi & \text{otherwise} \end{cases}. \tag{A.21}$$

Likewise, combining (A.17) and (A.20) delivers equilibrium inflation

$$\pi_t(\epsilon_{R,t}) = \begin{cases} \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\mu_\pi + \kappa\mu_c - \kappa\sigma_R\epsilon_{R,t}\Big] & \text{if } R_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi + \kappa\mu_c & \text{otherwise} \end{cases}. \tag{A.22}$$

If $X \sim N(\mu, \sigma^2)$ and C is a truncation constant, then

$$\mathbb{E}[X|X \geq C] = \mu + \frac{\sigma\phi_N(\alpha)}{1-\Phi_N(\alpha)},$$

where $\alpha = (C-\mu)/\sigma$, $\phi_N(x)$ and $\Phi_N(\alpha)$ are the probability density function (pdf) and the cumulative density function (cdf) of a $N(0,1)$. Define the cutoff value

$$C = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c. \tag{A.23}$$

Using the definition of a cdf and the formula for the mean of a truncated normal random variable, we obtain

$$\begin{aligned} \mathbb{P}[\epsilon_{R,t} \geq C/\sigma_R] &= 1 - \Phi_N(C_y/\sigma_R) \\ \mathbb{E}[\epsilon_{R,t} \,|\, \epsilon_{R,t} \geq C/\sigma_R] &= \frac{\sigma_R\phi_N(C/\sigma_R)}{1-\Phi_N(C/\sigma_R)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mu_c &= \frac{1-\Phi_N(C_y/\sigma_R)}{1+\kappa\psi}\Big[(1-\psi\beta)\mu_\pi + \mu_c\Big] - \frac{\sigma_R\phi_N(C_y/\sigma_R)}{(1+\kappa\psi)(1-\Phi_N(C_y/\sigma_R))} \qquad \text{(A.24)} \\ &\quad +\Phi_N(C_y/\sigma_R)\Big[\ln(r\pi_*) + \mu_c + \mu_\pi\Big] \\ \mu_\pi &= \frac{1-\Phi_N(C_y/\sigma_R)}{1+\kappa\psi}\Big[(\kappa+\beta)\mu_\pi + \kappa\mu_c\Big] - \frac{\kappa\sigma_R\phi_N(C_y/\sigma_R)}{(1+\kappa\psi)(1-\Phi_N(C_y/\sigma_R))} \qquad \text{(A.25)} \\ &\quad +\Phi_N(C_y/\sigma_R)\Big[\kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi + \kappa\mu_c\Big] \end{aligned}$$

The constants C, μ_c, and μ_π can be obtained by solving the system of nonlinear equations composed of (A.23) to (A.25).

B.2 Approximation Near the Deflation Steady State

Steady State. As before, let $\lambda = \nu(1-\beta)$. The steady-state nominal interest rate is $R_D = 1$, and provided that $\beta/(\gamma\pi_*) < 1$ and $\psi_1 > 1$:

$$\begin{aligned} r &= \gamma/\beta \\ \pi_D &= \beta/\gamma \\ c_D &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau} \\ y_D &= \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2\right]}. \end{aligned}$$

Log-Linearization. We omit the tildes from variables that capture deviations from the deflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price-setting equation (A.12) takes the form

$$\begin{aligned} 0 \;=\; & -\frac{\tau c_D^\tau}{\nu}c_t + \phi\beta\left[\left(1 - \frac{1}{2\nu}\right)\beta + \frac{\bar{\pi}}{2\nu}\right]\pi_t + \phi\beta(\beta - \bar{\pi})\left(1 - \frac{1}{2\nu}\right)\pi_t \\ & -\phi\beta^2(\beta - \bar{\pi})\Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}]\Big) - \phi\beta^3\mathbb{E}_t[\pi_{t+1}]. \end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\beta - \bar{\pi})^2}g_t - \frac{\phi\beta(\beta - \bar{\pi})}{1/g_* - \phi(\beta - \bar{\pi})^2}\pi_t$$

Finally, the monetary policy rule becomes

$$\begin{aligned} R_t \;=\; & \max\Big\{0,\; -(1-\rho_R)\ln(r\pi_*) - (1-\rho_R)\psi_1\ln(\pi_*/\beta) \\ & +(1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R\epsilon_{R,t}\Big\}. \end{aligned}$$

Approximate Piecewise-Linear Solution in Special Case. As for the approximate analysis of the targeted-inflation equilibrium, we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and

$\rho_g = 0$. In the deflation equilibrium, the steady-state inflation rate is $\pi_D = \beta$. To ease the expositions, we assume that the terms $|\pi_D - \bar{\pi}|$ that appear in the log-linearized equations above are negligible. Denote percentage deviations of a variable x_t from its deflation steady state by $\tilde{x}_t = \ln(x_t/x_D)$. If we let $\kappa_D = c_D/(\nu\phi\beta^2)$ and using the steady-state relationship $r = 1/\beta$

$$\begin{aligned} \tilde{R}_t &= \max\left\{0,\ -(\psi-1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma_R\epsilon_{R,t}\right\} \\ \tilde{c}_t &= \mathbb{E}_t[\tilde{c}_{t+1}] - (\tilde{R}_t - \mathbb{E}_t[\tilde{\pi}_{t+1}]) \\ \tilde{\pi}_t &= \beta\mathbb{E}_t[\tilde{\pi}_{t+1}] + \kappa_D\tilde{c}_t. \end{aligned} \tag{A.26}$$

Provided that $\psi > 1$, the ZLB is binding with high probability if the shock standard deviation σ_R is small. In this case, $\tilde{R}_t = 0$. An equilibrium in which all variables are *iid* can be obtained by adjusting the constants in (A.20) to (A.22):

$$\begin{aligned} \tilde{R}_t(\epsilon_{R,t}) &= \max\left\{0,\ \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi^D + \kappa\psi\mu_c^D - (\psi-1)\ln(r\pi_*) + \sigma_R\epsilon_{R,t}\right]\right\} \\ \tilde{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi^D + \mu_c^D + (\psi-1)\ln(r\pi_*) - \sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ \mu_c^D + \mu_\pi^D & \text{otherwise} \end{cases} \\ \tilde{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D + \kappa(\psi-1)\ln(r\pi_*) - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ (\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D & \text{otherwise} \end{cases}. \end{aligned} \tag{A.27}$$

In this simple model, the decision rules have a kink at the point in the state space where the two terms in the max operator of the interest rate equation are equal to each other. In the targeted-inflation equilibrium, this point in the state space is given by

$$\bar{\epsilon}_R^* = \frac{1}{\sigma_R}\left[-(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^* - \kappa\psi\mu_c^*\right],$$

whereas in the deflation equilibrium, it is

$$\bar{\epsilon}_R^D = \frac{1}{\sigma_R}\left[(\psi-1)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^D - \kappa\psi\mu_c^D\right],$$

Once $\epsilon_{R,t}$ falls below the threshold value $\bar{\epsilon}_R^*$ or $\bar{\epsilon}_R^D$, its marginal effect on the endogenous variables is zero. To the extent that $\bar{\epsilon}_R^D > 0 > \bar{\epsilon}_R^*$, it takes a positive shock in the deflation equilibrium to move away from the ZLB, whereas it takes a large negative monetary shock in the targeted-inflation equilibrium to hit the ZLB.

C Computational Details

C.1 Model Solution Algorithm

Algorithm 1 (Solution Algorithm) *1. Start with a guess for* Θ. *For the targeted-inflation equilibrium, this guess is obtained from a linear approximation around the inflation target. For the deflation equilibrium, it is obtained by assuming constant decision rules for inflation and* $\mathcal{E}$ *at the deflation steady state. For the sunspot equilibrium, it is obtained by letting the* $s_t = 1$ *decision rules come from the targeted-inflation equilibrium and the* $s_t = 0$ *decision rules come from the deflation equilibrium.*

2. *Given this guess, simulate the model for a large number of periods.*

3. *Given the simulated path, obtain the grid for the state variables over which the approximation needs to be accurate. Label these grid points as* $\{\mathcal{S}_1, ..., \mathcal{S}_M\}$. *For a fourth-order approximation, we use* $M = 130$. *For the targeted-inflation equilibrium, 79 of these grid points come from the ergodic distribution, obtained using a cluster-grid algorithm as in Judd, Maliar, and Maliar (2010). The remaining 51 come from the filtered exogenous state variables from 2000:Q1 to 2012:Q3. For the deflation equilibrium, we use a time-separated grid algorithm to deliver* 130 *points, which suits the behavior of this equilibrium better, since there are many periods when the economy is on the "edge" of the ergodic distribution at the ZLB. For the sunspot equilibrium, we use the same time-separated grid algorithm to deliver 156 points each for* $s_t = 1$ *and* $s_t = 0$, *and 312 points come from filtered states using multiple particles per period from the particle filter and over sampling the period 2009:Q2-2011:Q2.*

4. *Solve for the* Θ *by minimizing the sum of squared residuals obtained following the steps below using a variant of a Newton algorithm.*

   (a) *For a generic grid point* $\mathcal{S}_i$ *and the current value for* Θ, *compute* $f^1_\pi(\mathcal{S}_i; \Theta)$, $f^2_\pi(\mathcal{S}_i; \Theta)$, $f^1_\mathcal{E}(\mathcal{S}_i; \Theta)$, *and* $f^2_\mathcal{E}(\mathcal{S}_i; \Theta)$.

   (b) *Assume* $\zeta_i \equiv I\{R(\mathcal{S}_i, \Theta) > 1\} = 1$ *and compute* π_i, *and* $\mathcal{E}_i$, *as well as* y_i *and* c_i *using (23) and (24), substituting in (25).*

(c) *If R_i that follows from (25) using π_i and y_i obtained in (b) is greater than unity, then ζ_i is indeed equal to one. Otherwise, set $\zeta_i = 0$ (and thus $R_i = 1$) and recompute all other objects.*

(d) *The final step is to compute the residual functions. There are four residuals, corresponding to the four functions being approximated. For a given set of state variables $\mathcal{S}_i$, only two of them will be relevant since we either need the constrained decision rules or the unconstrained ones. The residual functions will be given by*

$$\mathcal{R}^1(\mathcal{S}_i) = \mathcal{E}_i - \left[\int\int\int \frac{c(\mathcal{S}')^{-\tau}}{\gamma z'\pi(\mathcal{S}')} dF(z')\, dF(g')\, dF(\epsilon_R')\right] \tag{A.28}$$

$$\mathcal{R}^2(\mathcal{S}_i) = f(c_i, \pi_i, y_i) - \phi\beta \int\int\int c(\mathcal{S}')^{-\tau} y(\mathcal{S}') \left[\pi(\mathcal{S}') - \bar{\pi}\right] \pi(\mathcal{S}') dF(z')\, dF(g')\, dF(\epsilon_R') \tag{A.29}$$

Note that this step involves computing $\pi(\mathcal{S}')$, $y(\mathcal{S}')$, $c(\mathcal{S}')$, and $R(\mathcal{S}')$ which is done following steps (a)-(c) above for each value of $\mathcal{S}'$. We use a non product monomial integration rule to evaluate these integrals.

(e) *The objective function to be minimized is the sum of squared residuals obtained in (d).*

5. *Repeat steps 2-4 a sufficient number of times so that the ergodic distribution remains unchanged from one iteration to the next. For the targeted-inflation equilibrium and the sunspot equilibrium, we also iterate between solution and filtering to make sure the filtered states used in the solution grid remain unchanged.*

We start our solution from a second-order approximation and move to a third- and fourth-order approximation by using the previous solution. We use analytical derivatives of the objection function, which speeds up the solution by two orders of magnitude. As a measure of accuracy, we compute the approximation errors from A.28 and A.29, converted to consumption units. For the targeted-inflation equilibrium, these are in the order of 10^{-6}. For the deflation and sunspot equilibria, they are higher at 10^{-4} and 10^{-5}, respectively, but still very reasonable given the complexity of the model.

Figure A-1: Solution Grid for the Targeted-Inflation Equilibrium

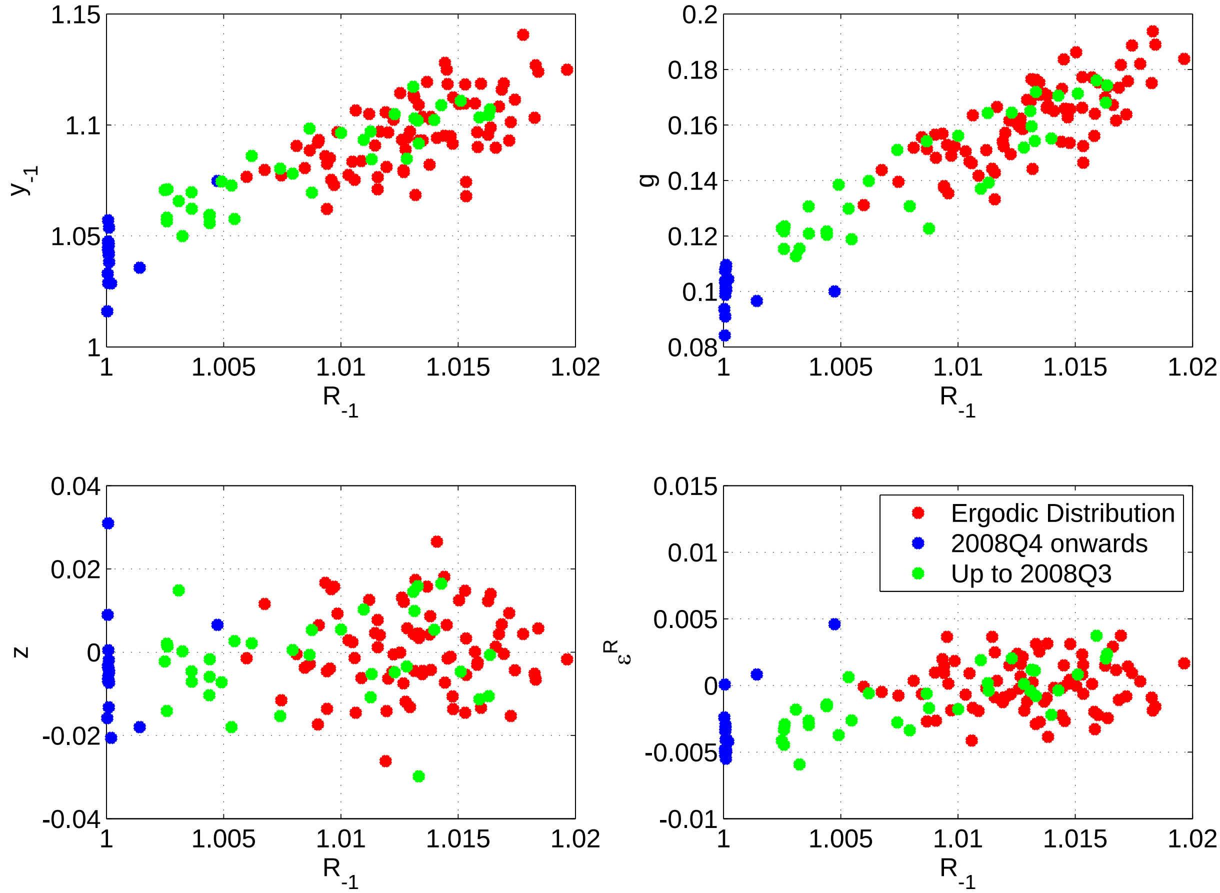

Figure A-1 shows the solution grid for the targeted-inflation equilibrium. For each panel, we have R_{t-1} on the x axis and the other state variables on the y axis. The red dots are the grid points that represent the ergodic distribution, the green points are the filtered states from 2000:Q1 to 2008:Q3, and the blue points are the filtered state for the period after 2008:Q3. It is evident that the filtered states lie in the tails of the ergodic distribution of the targeted-inflation equilibrium, which assigns negligible probability to zero interest rates and the exogenous states that push interest rates toward the ZLB. By adding these filtered states to the grid points, we ensure that our approximation will be accurate in these low-probability regions.

C.2 Details of Policy Experiments

Algorithm 2 (Effect of Combined Fiscal and Monetary Policy Intervention) *For $j = 1$ to $j = n_{sim}$, repeat the following steps:*

1. *Initialize the simulation by setting $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ equal to the mean estimate obtained with the particle filter.*

2. *Generate baseline trajectories based on the innovation sequence $\{\epsilon_t^{(j)}\}_{t=1}^H$ by letting $[\epsilon_{z,t}^{(j)}, \epsilon_{g,t}^{(j)}]' \sim N(0, I)$ and setting $\epsilon_{R,t} = 0$.*

3. *Generate the innovation sequence for the counterfactual trajectories according to*

$$\begin{aligned}
\epsilon_{g,1}^{I(j)} &= \delta^{ARRA} + \epsilon_{g,1}^{(j)}; \quad \epsilon_{g,t}^{I(j)} = \epsilon_{g,t}^{(j)} \quad \textit{for} \quad t = 2, \dots, H; \\
\epsilon_{z,t}^{I(j)} &= \epsilon_{z,t}^{(j)} \quad \textit{for} \quad t = 1, \dots, H; \\
\epsilon_{R,t}^{I(j)} &= \epsilon_{R,t}^{(j)} = 0 \quad \textit{for} \quad t = 9, \dots, H;
\end{aligned}$$

   In periods $t = 1, \dots, 8$, conditional on $\{\epsilon_{g,t}^{I(j)}, \epsilon_{z,t}^{I(j)}\}_{t=1}^4$, determine $\epsilon_{R,t}^{I(j)}$ by solving for the smallest $\tilde{\epsilon}_{R,t}$ such that it is less than $2\sigma_R$ in absolute value, that yields either

$$R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = \tilde{\epsilon}_{R,t}) = 1 \quad \textit{or} \quad 400 \ln \left(R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = 0) - R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = \tilde{\epsilon}_{R,t}) \right) = 1.$$

4. *Conditional on $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$, compute $\{R_t^{(j)}, y_t^{(j)}, \pi_t^{(j)}\}_{t=1}^H$ and $\{R_t^{I(j)}, y_t^{I(j)}, \pi_t^{I(j)}\}_{t=1}^H$ based on $\{\epsilon_t^{(j)}\}$ and $\{\epsilon_t^{I(j)}\}$, respectively, and let*

$$IRF^{(j)}(x_t | \epsilon_{g,1}, \epsilon_{R,1:8}) = (\ln x_t^{I(j)} - \ln x_t^{(j)}). \tag{A.30}$$

Compute medians and percentile bands based on $IRF^{(j)}(x_t | \epsilon_{g,1}, \epsilon_{R,1:8})$, $j = 1, \dots, n_{sim}$. □

When we consider only a fiscal policy, we set $\epsilon_{R,t}^{I(j)} = 0$ for $t = 1, ..., 8$ as well.

D Estimation of Second-Order Approximated DSGE Model

Table A-1 summarizes the prior and posterior distribution from the Bayesian estimation of the second-order approximated version of the DSGE model. The estimation sample is 1984:Q1 to 2007:Q4. The parameter ϕ that is used in the main text is related to the parameter κ (Phillips curve slope of a linearized version of the DSGE model) according to $\phi = \frac{\tau(1-\nu)}{(\nu\pi^2\kappa)}$. The parameters r^*, π^*, and γ are fixed at the sample means of the ex-post real rate, the inflation rate, and output growth. We assume that $\bar{\pi} = 1$, meaning that any price change is costly.

Table A-1: Posterior Estimates for DSGE Model Parameters

		Prior		Posterior	
Parameter	Density	Para 1	Para 2	Mean	90% Interval
τ	Gamma	2.00	0.25	1.50	[1.14, 1.89]
κ	Gamma	0.30	0.10	0.17	[0.05, 0.30]
ψ_1	Gamma	1.50	0.10	1.36	[1.27, 1.43]
ρ_r	Beta	0.50	0.20	0.64	[0.55, 0.72]
ρ_g	Beta	0.80	0.10	0.86	[0.82, 0.91]
ρ_z	Beta	0.20	0.10	0.11	[0.03, 0.24]
$100\sigma_r$	Inv Gamma	0.30	4.00	0.21	[0.17, 0.26]
$100\sigma_g$	Inv Gamma	0.40	4.00	0.78	[0.66, 0.93]
$100\sigma_z$	Inv Gamma	0.40	4.00	1.03	[0.83, 1.32]
$400(r^* - 1)$	Fixed	2.78			
$400(\pi^* - 1)$	Fixed	2.52			
$100(\gamma - 1)$	Fixed	0.48			
$\overline{\pi}$	Fixed	1.00			
ψ_2	Fixed	0.80			
ν	Fixed	0.10			
$\frac{1}{g}$	Fixed	0.85			

Notes: Para (1) and Para (2) list the means and the standard deviations for Beta and Gamma; and s and ν for the Inverse Gamma distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1} e^{-\nu s^2/2\sigma^2}$. The effective prior is truncated at the boundary of the determinacy region. Estimation sample is 1984:Q1 to 2007:Q4. As 90% credible interval, we are reporting the 5th and 95th percentile of the posterior distribution.

E Particle Filter

The particle filter is used to extract information about the state variables of the model from data on output growth, inflation, and nominal interest rates over the period 2000:Q1 to 2012:Q3.

E.1 State-Space Representation

Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables, which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$, is the Markov-switching process.

$$y_t = \Psi(x_t) + \nu_t \tag{A.31}$$

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{A.32}$$

$$x_t = F_{s_t}(x_{t-1}, \epsilon_t) \tag{A.33}$$

The first equation is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents the law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. We subsequently use the densities $p(y_t|x_t)$, $p(s_t|s_{t-1})$, and $p(x_t|x_{t-1}, s_t)$ to summarize the measurement and the state transition equations. The targeted-inflation equilibrium yields a state-space system that is a special case: the discrete state s_t is constant.

E.2 Sequential Importance Sampling Approximation

Let $z_t = [x_t', s_t]'$ and $Y_{t_0:t_1} = \{y_{t_0}, \ldots, y_{t_1}\}$. Particle filtering relies on sequential importance sampling approximations. The distribution $p(z_{t-1}|Y_{1:t-1})$ is approximated by a set of pairs

$\{(z_{t-1}^{(i)}, \pi_{t-1}^{(i)})\}_{i=1}^N$ in the sense that

$$\frac{1}{N}\sum_{i=1}^{N} f(z_{t-1}^{(i)})\pi_{t-1}^{(i)} \xrightarrow{a.s.} \mathbb{E}[f(z_{t-1})|Y_{1:t-1}], \tag{A.34}$$

where $z_{t-1}^{(i)}$ is the i'th particle, $\pi_{t-1}^{(i)}$ is its weight, and N is the number of particles. An important step in the filtering algorithm is to draw a new set of particles for period t. In general, these particles are drawn from a distribution with a density that is proportional to $g(z_t|Y_{1:t}, z_{t-1}^{(i)})$, which may depend on the particle value in period $t-1$ as well as the observation y_t in period t. This procedure leads to an importance sampling approximation of the form:

$$\begin{aligned}
\mathbb{E}[f(z_t)|Y_{1:t}] &= \int_{z_t} f(z_t)\frac{p(y_t|z_t)p(z_t|Y_{1:t-1})}{p(y_t|Y_{1:t-1})}dz_t \qquad \text{(A.35)}\\
&= \int_{z_{t-1:t}} f(z_t)\frac{p(y_t|z_t)p(z_t|z_{t-1})p(z_{t-1}|Y_{1:t-1})}{p(y_t|Y_{1:t-1})}dz_{t-1:t}\\
&\approx \frac{\frac{1}{N}\sum_{i=1}^N f(z_t^{(i)})\frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{g(z_t^{(i)}|Y_{1:t},z_{t-1}^{(i)})}\pi_{t-1}^{(i)}}{\frac{1}{N}\sum_{j=1}^N \frac{p(y_t|z_t^{(j)})p(z_t^{(j)}|z_{t-1}^{(j)})}{g(z_t^{(j)}|Y_{1:T},z_{t-1}^{(i)})}\pi_{t-1}^{(i)}}\\
&= \frac{1}{N}\sum_{i=1}^{N} f(z_t^{(i)})\left(\frac{\tilde{\pi}_t^{(i)}}{\frac{1}{N}\sum_{j=1}^N \tilde{\pi}_t^{(j)}}\right) = \frac{1}{N}\sum_{i=1}^{N} f(z_t^{(i)})\pi_t^{(i)},
\end{aligned}$$

where the unnormalized and normalized probability weights are given by

$$\tilde{\pi}_t^{(i)} = \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{g(z_t^{(i)}|Y_{1:T}, z_{t-1}^{(i)})}\pi_{t-1}^{(i)} \quad \text{and} \quad \pi_t^{(i)} = \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^N \tilde{\pi}_t^{(j)}}, \tag{A.36}$$

respectively. In simple versions of the particle filter, $z_t^{(i)}$ is often generated by simulating the model forward, which means that $g(z_t^{(i)}|Y_{1:T}, z_{t-1}^{(i)}) \propto p(z_t^{(i)}|z_{t-1}^{(i)})$, and the formula for the particle weights simplifies considerably. Unfortunately, this approach is quite inefficient in our application, and we require a more elaborate density $g(\cdot|\cdot)$ described below that accounts for information in y_t. The resulting extension of the particle filter is known as auxiliary particle filter, e.g. Pitt and Shephard (1999).

E.3 Filtering

Initialization. To generate the initial set of particles $\{(z_0^{(i)}, \pi_0^{(i)}\}_{i=1}^N$, for each i, simulate the DSGE model for T_0 periods, starting from the targeted-inflation steady state, and set

$\pi_0^{(i)} = 1$.

Sequential Importance Sampling. For $t = 1$ to T:

1. $\{z_{t-1}^{(i)}, \pi_{t-1}^{(i)}\}_{i=1}^N$ is the particle approximation of $p(z_{t-1}|Y_{1:t-1})$. For $i = 1$ to N:

   (a) Draw $z_t^{(i)}$ conditional on $z_{t-1}^{(i)}$ from $g(z_t|Y_{1:t}, z_{t-1}^{(i)})$.

   (b) Compute the unnormalized particle weights $\tilde{\pi}_t^{(i)}$ according to (A.36).

2. Compute the normalized particle weights $\pi_t^{(i)}$ and the effective sample size $ESS_t = N^2/\sum_{i=1}^N (\pi_t^{(i)})^2$.

3. Resample the particles via deterministic resampling (see Kitagawa (1996)). Reset weights to be $\pi_t^{(i)} = 1$ and approximate $p(z_t|Y_{1:t})$ by $\{(z_t^{(i)}, \pi_t^{(i)})\}_{i=1}^n$.

E.4 Tuning of the Filter

In the empirical analysis, we set $T_0 = 50$ and $N = 500,000$. We also fix the measurement error standard deviations for output growth, inflation, and interest rates at 0.1, respectively. Since our model has discrete and continuous state variables, we write

$$p(z_t|z_{t-1}) = \begin{cases} p_0(x_t|x_{t-1}, s_t = 0)\mathbb{P}\{s_t = 0|s_{t-1}\} & \text{if } s_t = 0 \\ p_1(x_t|x_{t-1}, s_t = 1)\mathbb{P}\{s_t = 1|s_{t-1}\} & \text{if } s_t = 1 \end{cases}$$

and consider proposal densities of the form

$$q(z_t|z_{t-1}, y_t) = \begin{cases} q_0(x_t|x_{t-1}, y_t, s_t = 0)\lambda(z_{t-1}, y_t) & \text{if } s_t = 0 \\ q_1(x_t|x_{t-1}, y_t, s_t = 1)(1 - \lambda(z_{t-1}, y_t)) & \text{if } s_t = 1 \end{cases},$$

where $\lambda(x_{t-1}, y_t)$ is the probability that $s_t = 0$ under the proposal distribution. We use $q(\cdot)$ instead of $g(\cdot)$ to indicate that the densities are normalized to integrate to one.

We effectively generate draws from the proposal density through forward iteration of the state transition equation. To adapt the proposal density to the observation y_t, we draw $\epsilon_t^{(i)} \sim N(\mu^{(i)}, \Sigma^{(i)})$ instead of the model-implied $\epsilon_t \sim N(0, I)$. In slight abuse of notation (ignoring that the dimension of x_t is larger than the dimension of ϵ_t and that its distribution is singular), we can apply the change of variable formula to obtain a representation of the proposal density

$$q(x_t^{(i)}|x_{t-1}^{(i)}) = q_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

Using the same change-of-variable formula, we can represent

$$p(x_t^{(i)}|x_{t-1}^{(i)}) = p_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

By construction, the Jacobian terms cancel and the ratio that is needed to calculate the unnormalized particle weights for period t in (A.36) simplifies to

$$\tilde{\pi}_t^{(i)} = p(y_t|z_t^{(i)}) \frac{\exp\left\{-\frac{1}{2}\epsilon_t^{(i)\prime}\epsilon_t^{(i)}\right\}}{|\Sigma_\epsilon^{(i)}|^{-1/2}\exp\left\{-\frac{1}{2}(\epsilon_t^{(i)} - \mu^{(i)})'[\Sigma^{(i)}]^{-1}(\epsilon_t^{(i)} - \mu^{(i)})\right\}} \pi_{t-1}^{(i)}.$$

The choice of μ and Σ is described below.

Targeted-Inflation Equilibrium. Since the discrete state s_t is irrelevant in this equilibrium, let $z_t = x_t$. We break the sample period into two parts: 2000:Q1 to 2008:Q4 and 2009:Q1 to 2012:Q3. In the second period, the economy was at the ZLB and the filter requires a different proposal density.

For the first part of the sample, we run the Kalman filter for the log-linearized version of the DSGE model in parallel with the particle filter and set $\mu^{(i)} = \epsilon_{t|t}^{(i)}$ and $\Sigma^{(i)} = P_{t|t}^{(i)}$, which are respectively the mean and variance of ϵ_t conditional on $Y_{1:t}$ and $z_{t-1} = z_{t-1}^{(i)}$. For the second part of the sample, the Kalman-filtered shocks become very inaccurate because the log-linearized DSGE model misses the ZLB. Instead, we let $z_{t-1|t-1}$ be a particle filter approximation of $\mathbb{E}[z_{t-1}|Y_{1:t-1}]$ and define

$$\bar{\pi}_t(\epsilon_t) = p(y_t|F(z_{t-1|t-1}, \epsilon_t)) \exp\left\{-\frac{1}{2}\epsilon_t'\epsilon_t\right\} |\Sigma_\epsilon|^{1/2}\pi_{t-1}^{(i)}.$$

We use a grid search over ϵ_t to determine a value $\bar{\epsilon}$ that maximizes this objective function and then set $\mu^{(i)} = \bar{\epsilon}$. (Executing the grid search conditional on each $z_{t-1}^{(i)}$, $i = 1, \ldots, N$ turned out to be too time consuming.)

Sunspot Equilibrium. The filter is initialized by simulating the model for $T_0 = 50$ periods conditional on $s_t = 1$. For the period of 2000:Q1 to 2008:Q4, we use the simple grid search approach described in the previous paragraph to generate shocks under which we simulate the state-transition equation forward. Starting in 2008:Q4, we use the information from the grid search to construct a mixture-of-normals proposal distribution for $\epsilon_t^{(i)}$. While more time consuming, this mixture proposal improves the accuracy of the particle filter. At each iteration, we conduct separate computations for $s_t = 0$ and $s_t = 1$. We then compute the posterior odds of $s_t = 0$ and $s_t = 1$ and select the regime-conditional particles accordingly. For the ex-ante policy analysis, we run the filter from 2009:Q1 onward conditional on a sequence of regimes for the periods from 2009:Q2 to 2011:Q1.

F Calibration of the Policy Experiment

Table A-2 summarizes the award and disbursements of funds for federal contracts, grants, and loans. We translate the numbers in the table into a one-period location shift of the distribution of $\epsilon_{g,t}$. In our model, total government spending is a fraction ζ_t of aggregate output, where ζ_t evolves according to an exogenous process:

$$G_t = \zeta_t Y_t; \quad \zeta_t = 1 - \frac{1}{g_t}; \quad \ln(g_t/g_*) = \rho_g \ln(g_{t-1}/g_*) + \sigma_g \epsilon_{g,t}$$

For the subsequent calibration of the fiscal intervention, it is convenient to define the percentage deviations of g_t and ζ_t from their respective steady states: $\hat{g}_t = \ln(g_t/g_*)$ and $\hat{\zeta}_t = \ln(\zeta_t/\zeta_*)$. According to the parameterization of the DSGE model in Table 1, $\zeta_* = 0.15$ and $g_* = 1.177$. Thus, government spending is approximately 15% of GDP. We assume that the fiscal expansion approximately shifts $\hat{\zeta}_t$ to $\hat{\zeta}_t^I = \hat{\zeta}_t + \hat{\zeta}_t^{ARRA}$.

We construct $\hat{\zeta}_t^{ARRA}$ as follows. Let G_t^{ARRA} correspond to the additional government spending stipulated by ARRA. Since we focus on received rather than awarded funds, G_t^{ARRA} corresponds to the third column of Table A-2. The size of the fiscal expansion as a fraction of GDP is

$$\zeta_t^{ARRA} = G_t^{ARRA}/Y_t,$$

where Y_t here corresponds to the GDP data reported in the last column of Table A-2. We then divide by ζ_* to convert it into deviations from the steady-state level: $\hat{\zeta}_t^{ARRA} = \zeta_t^{ARRA}/\zeta_*$.

Taking a log-linear approximation of the relationship between g_t and ζ_t leads to

$$\hat{g}_t^{ARRA} = 0.177 \cdot G_t^{ARRA}/(\zeta_* Y_t).$$

In Figure A-2, we compare $\hat{g}_t^{ARRA}$ constructed from the data in Table A-2 to $(\hat{g}_t^I - \hat{g}_t)$, where $\delta^{ARRA} = 0.011$.[14] While the actual path of the received funds is not perfectly monotone, the calibrated intervention in the DSGE model roughly matches the actual intervention both in terms of magnitude and decay rate.

Table A-2: ARRA Funds for Contracts, Grant, and Loans

	Awarded	Received	Nominal GDP
2009:3	158	36	3488
2009:4	17	18	3533
2010:1	26	8	3568
2010:2	16	24	3603
2010:3	33	26	3644
2010:4	9	21	3684
2011:1	4	19	3704
2011:2	4	20	3751
2011:3	8	17	3791
2011:4	0	12	3830
2012:1	3	9	3870
2012:2	0	8	3899

Notes: Data were obtained from www.recovery.org.

[14]Recall that $\sigma_g = 0.0078$.

Figure A-2: Calibration of Fiscal Policy Intervention